Math 151 Introduction to Eigenvectors

The motivating example we used to describe matrixes was landscape change and vegetation succession. We chose the simple example of Bare Soil (B), being replaced by Grasses (G) and then these being replaced by Shrubs. Describing a landscape at a particular time we saw could be done using a vector that gave the area in km^2 (or fraction of total area which is then a number between 0 and 1) as a vector with three entries (B G S) giving the fraction of the landscape of each species. If these were fraction of area, then the vector of numbers would sum to one. If these were area in each successional type, then they would sum to the total area on the landscape.

We next described how this vector v = (B G S) could change through time due to the process of succession, and noted that this can be described mathematically by multiplying a vector times a matrix which specifies over the underlying time period (which could be a decade) how much of each type changes to each other type. So if the matrix is

$$M = \begin{bmatrix} .4 & 0 & 0 \\ .5 & .8 & 0 \\ .1 & .2 & 1 \end{bmatrix}$$

then this says that each decade 40% of the bare soil stays bare soil, 50% changes to grass and 10% changes to shrub. It also says that 80% of the grass area stays grass and 20% becomes shrub in the course of a decade. If an area becomes shrub, then it stays shrub.

If we start out with a certain area in each successional stage, then we can use this matrix to project forward in time one decade by multiplying the matrix times the vector for the current stage distribution. That is, if y_t gives the vector of area in each stage at decade t, then

$$y_{t+1} = M\, y_t$$

This is called a matrix equation since it is an equation that has a matrix in it and it allows us to project from one decade forward to the next decade.

So for example, if we start out with 100 km^2 in a region that is bare soil, and no grass or shrub, then

$$y_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

gives the initial distribution of stages at time 0 and we can get the distribution of stages at time 1 decade using

$$y_1 = M\, y_0$$

or

$$y_1 = \begin{bmatrix} .4 & 0 & 0 \\ .5 & .8 & 0 \\ .1 & .2 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 50 \\ 10 \end{bmatrix}$$

so that after one decade this region has 40 km^2 in bare soil, 50 km^2 in grass and 10 km^2 in shrub.

Doing this again we can get how much of the landscape is in each time at decade 2 from

$$y_2 = M\, y_1 = M^2\, y_0 = \begin{bmatrix} .4 & 0 & 0 \\ .5 & .8 & 0 \\ .1 & .2 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 16 \\ 60 \\ 24 \end{bmatrix}$$

We can continue to multiply the matrix M times each decades vector to get the next decades vector. Look at the entries in the vector y_2 and note that they sum to 100. This is because there is no land craeted or destroyed here - each unit of area must remain as one of the three types B, G or S. What do you think will happen if we continue to find Y_t for larger and larger values of t? Yes, you are correct, eveything eventually becomes shrub, and so this means that the vector y_t gets closer and closer to the vector

$$\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

We will see later in the course that the mathematical way to state this is

$$\lim_{t\to\infty} y_t = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

This is a fancy way of saying that eventually all the area becomes shrub. In this case it was pretty easy to intuit what would happen in the long-term in this landscape. This would not be as easy to determine if we modify the situation as we mentioned earlier to look at the effect of fire (fire is only one kind of disturbance in this system that could cause the system to switch back to bare soil - other forms of distubance in natural systems that could cause similar effects are windstorms, hurricanes, and disease. So our objective in this section of the course is to develop a mathematical way to see what happens after a long-time in a structured system. We are using the example of succession, but we'll also see that exactly the same methods work to determine the long-term fraction os a population in each of a set of age ranges, which is part of the area of demography we have already mentioned.

Note that if we start out a landscape with 100 km^2 in shrub, then after a decade we get

$$\begin{bmatrix} .4 & 0 & 0 \\ .5 & .8 & 0 \\ .1 & .2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

which means the landscape doesn't change at all over the decade - everything remains in the shrub stage. We call the vector

$$\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

an eigenvector for the matrix M because if we start at time 0 at this eigenvector of distributions of stages, we stay there forever. This is an equilibrium state for the system of succession - once there we stay there forever. More than that, in this situation we can show that no matter what distribution of initial states we start at, the system eventually approaches this eigenvector - in this

case we say it is stable.

In general, we will say that v is an eigenvector for a matrix A if thee is some constant λ so that

$$\lambda \, v = A \, v$$

and we say that v has associated eigenvalue λ. in the above situation, the vector

$$\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

is an eigenvector with eigenvalue $\lambda = 1$.

If we were to change the matrix M to include the effect of fire, one case we discussed is

$$N = \begin{bmatrix} .4 & .1 & .05 \\ .5 & .7 & 0 \\ .1 & .2 & .95 \end{bmatrix}$$

If we start out with an initial distribution y_0 , then we can see by iterating that

$$y_t = N^t \, y_0$$

For example, if we start out with 100 km^2 in bare soil then using Matlab as a tool to calculate

$$y_5 = N^5 \, y_0 = \begin{bmatrix} 8 \\ 9 \\ 83 \end{bmatrix}$$

and

$$y_{100} = N^{100} \, y_0 = \begin{bmatrix} 8.8 \\ 14.7 \\ 76.5 \end{bmatrix}$$

So that after a long time the vector of states approaches

$$v = \begin{bmatrix} 8.8 \\ 14.7 \\ 76.5 \end{bmatrix}$$

and if we start out at this state we see that

$$\begin{bmatrix} 8.8 \\ 14.7 \\ 76.5 \end{bmatrix} = \begin{bmatrix} .4 & .1 & .05 \\ .5 & .7 & 0 \\ .1 & .2 & .95 \end{bmatrix} \begin{bmatrix} 8.8 \\ 14.7 \\ 76.5 \end{bmatrix}$$

so that v is an eigenvector (and its eigenvalue is one).

There are a variety of methods to find eigenvectors and eigenvalues. One way to find an eigenvector is numerically. We can use Matlab to find for any matrix P, the matrix P^n, where n is a large number (say 100), then multiply this times the initial vector for the landscape, y_0, to get a numerical answer for the long-term state of the landscape. If the initial vector y_0 contained the

fraction of the landscape in each vegetation type, then $P^{100}\ y_0$ will be a vector giving the long-term fraction of the landscape in each vegetation type. If the initial vector y_0 rather represents the number of hectares or acres of each type, then divide each term in $P^{100}\ y_0$ by the sum of the components of this vector to get the long-term fraction in each state (the eigenvector is specified only up to a constant multiple).

A second way to get the eigenvector is to realize that it arises when the long term structure of the system doesn't change. This is expressed as $P\ y = \lambda y$ where λ is a constant that represents how the vector of vegetation types increases or decreases through one time period. In our case of a fixed landscape, land area is neither created nor destroyed, so there is no change from one time period to another and so $\lambda = 1$. This means to find the eigenvector all we need to do is find a vector y that satisfies $P\ y = y$. This is easy to do using simple algebra for small matrices, but for larger ones the mathematics becomes more difficult. In this case you either use the theory of determinants, or else use Matlab to find the answer.

In this class, we expect that you will be able to compute by hand the eigenvectors and eigenvalues only for the simplest matrices - 2x2 ones. Consider the 2x2 matrix

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and lets look at the equations arising from

$$P\ y = \lambda\ y$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

so

$$P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Then this holds if $a\ y_1 + b\ y_2 = \lambda\ y_1$ and $c\ y_1 + d\ y_2 = \lambda\ y_2$. In order for these to both hold, we need

$$y_1 = \frac{b}{\lambda - a}\ y_2$$

and

$$y_1 = \frac{\lambda - d}{c}\ y_2$$

So the only way these can both hold is if $y_1 = y_2 = 0$ (which is not an interesting case) or if $\frac{b}{\lambda - a} = \frac{\lambda - d}{c}$. This is a quadratic equation in λ:

$$\lambda^2 - (a + d)\ \lambda + a\ d - b\ c = 0$$

and this equation is called the characteristic equation for this matrix. We call $a + d$ the Trace of the matrix P (Tr(P)) and $a\ d - b\ c$ the Determinant of the matrix P (Det(P)). By solving this quadratic for the roots, we find the eigenvalues λ (there will be two in general for a 2x2 matrix).

To find the eigenvector, we plug in one of these λ values to find the ratio of y_1 to y_2, and this gives us the eigenvector up to a constant.

Note that in the case of our succession model where the total area of the landscape doesn't change, we have $\lambda = 1$, we must have $1 - (a + d) + a\, d - b\, c = 0$. The landscape transition matrix can be written in this case as

$$P = \begin{bmatrix} a & b \\ 1 - a & 1 - b \end{bmatrix}$$

because the columns of the matrix P sum to one (so $a + c = 1$ and $b + d = 1$). So Tr(P) = $a + 1 - b$ and Det(P) = $a (1 - b) - b (1 - b)$ and in this case then

$$y_1 = \frac{b}{1 - a}\, y_2 = \frac{1 - d}{c}\, y_2$$

gives the eigenvector. Normalize this so they sum to 1.

As an example, consider the matrix

$$P = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

then Tr(P) = 6 and Det(P) = 5 so the chaarcteristic equation is

$$\lambda^2 - 6\, \lambda + 5 = 0$$

so the eigenvalues are $\lambda = 1$ and $\lambda = 5$ with eigenvector

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

for $\lambda = 1$ and eigenvector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda = 5$. Note that only the eigenvector for $\lambda = 5$ can be normalized, since the eigenvector for $\lambda = 1$ has a negative element. For $\lambda = 5$, the sum of the elements of the eigenvector is 2 so divide each of the elements by 2 to get the normalized eigenvector for $\lambda = 5$ which is

$$\begin{bmatrix} .5 \\ .5 \end{bmatrix}$$