Exponentials, logarithms and rescaling of data

Math 151 : Sept 9, 2003

Scott Sylvester substituting for Lou Gross

From last class

- Correlation coefficient
- $-1 \leq \rho \leq 1$, always

$$\rho = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

- $f(x) = a^x$, exponential
- $f(x) = \log_a x$, logarithm
- $f(x) = ax^b$, allometric function

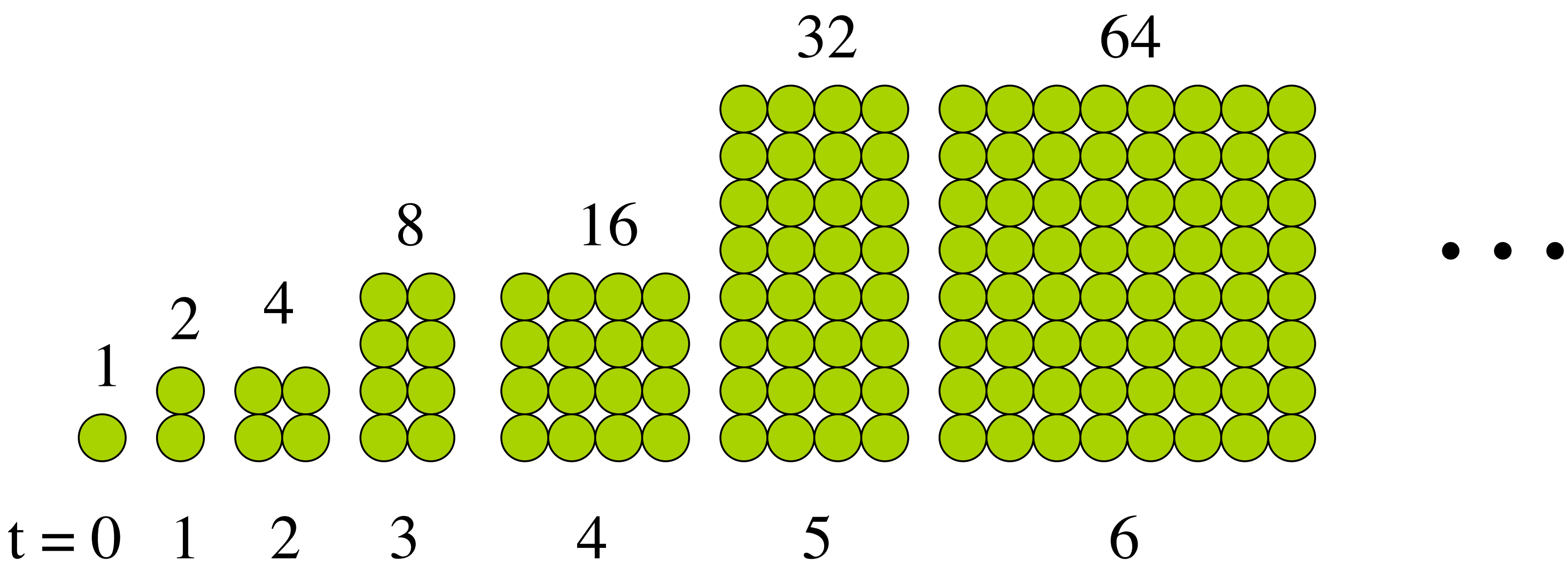
Motivation

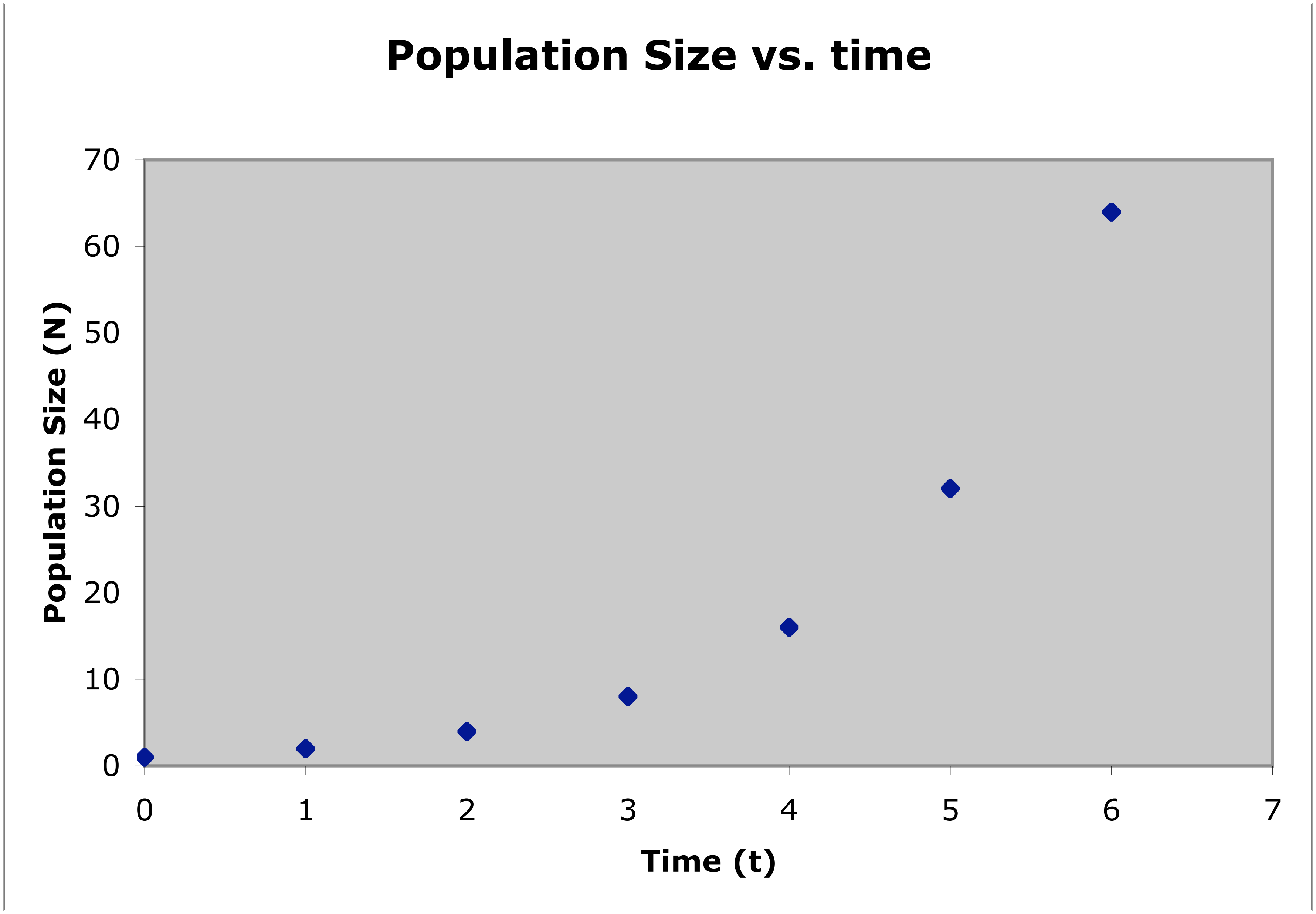
- There are lots of non-linear phenomena in the world:
  - Population growth
  - Relationship between different parts/aspects of an organism (allometric relationships)
  - The number of species found in a given area (species-area relationships)
  - Radio active decay
  - Many others
- As it turns out exponentials, logs and allometric functions are useful in understanding these phenomena

- Population growth is a classic example

  Algae : cell division

  Geometric growth

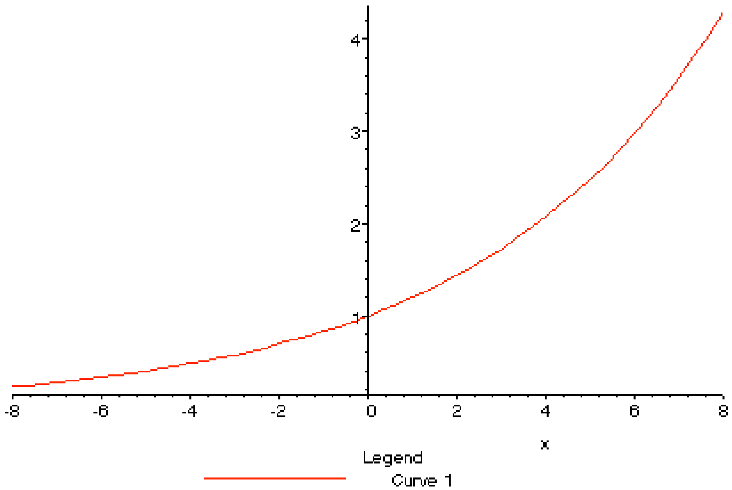


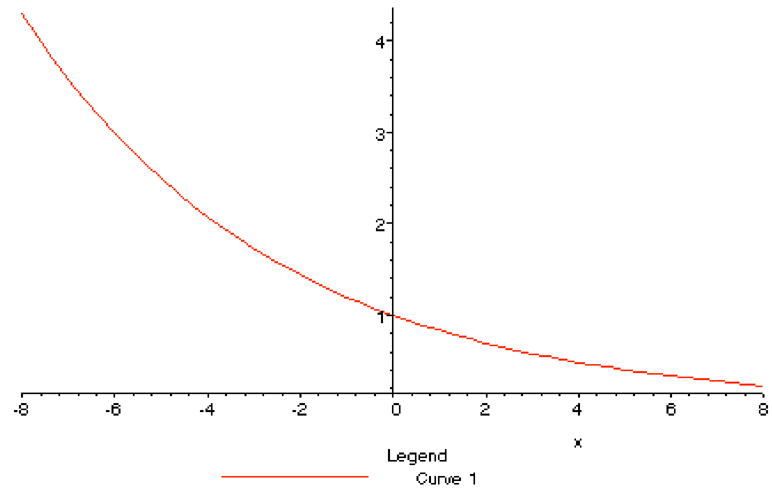


Population Size vs. time
Population Size (N)
70
60
50
40
30
20
10
0
0
1
2
3
4
5
6
7
Time (t)

Exponentials

$f(x) = a^x$, $a > 0$





$a > 1$

exponential increase

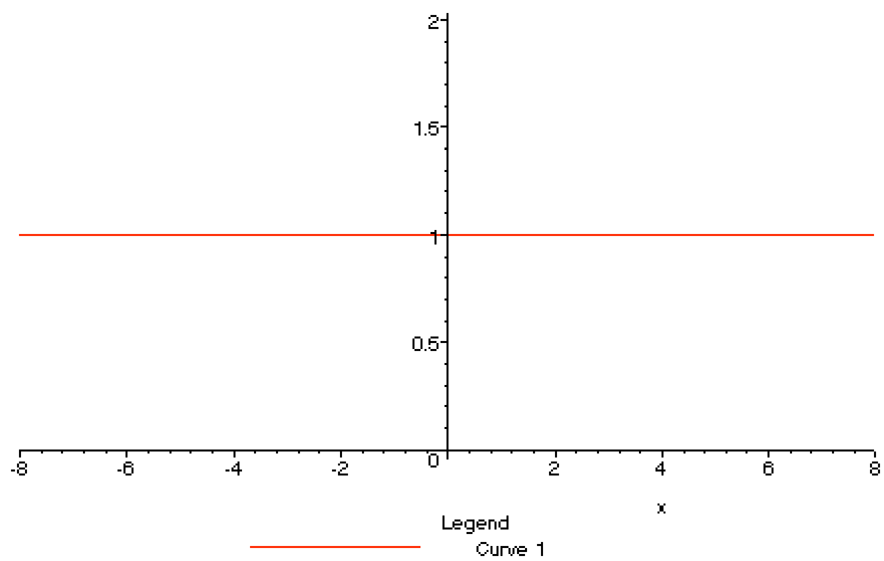
$$\lim_{x \to -\infty} ax = 0$$

$0 < a < 1$

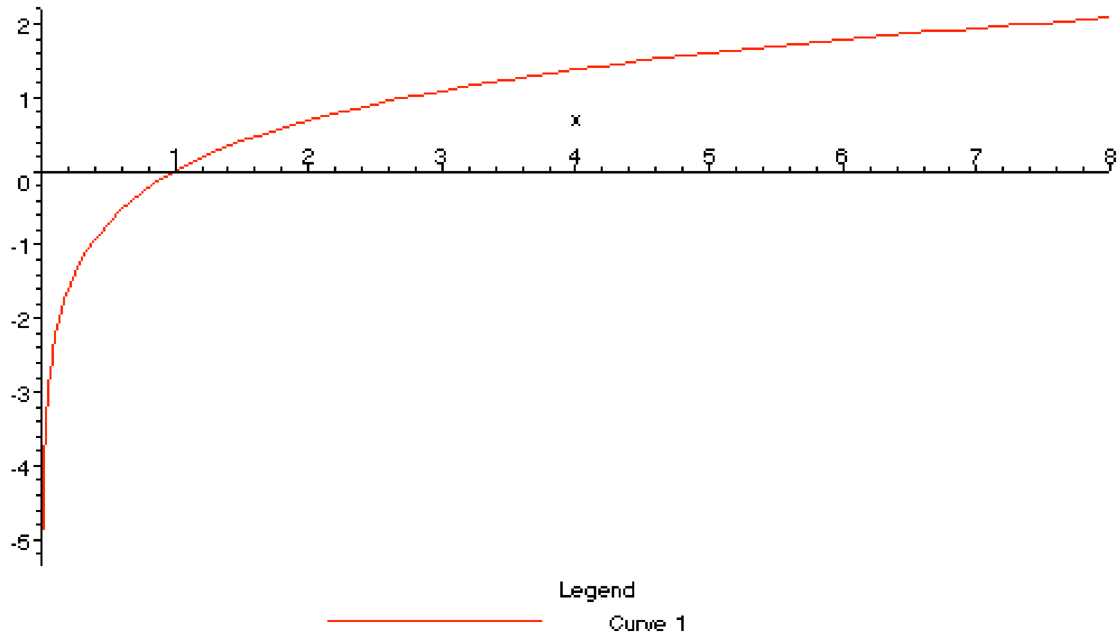
exponential decrease

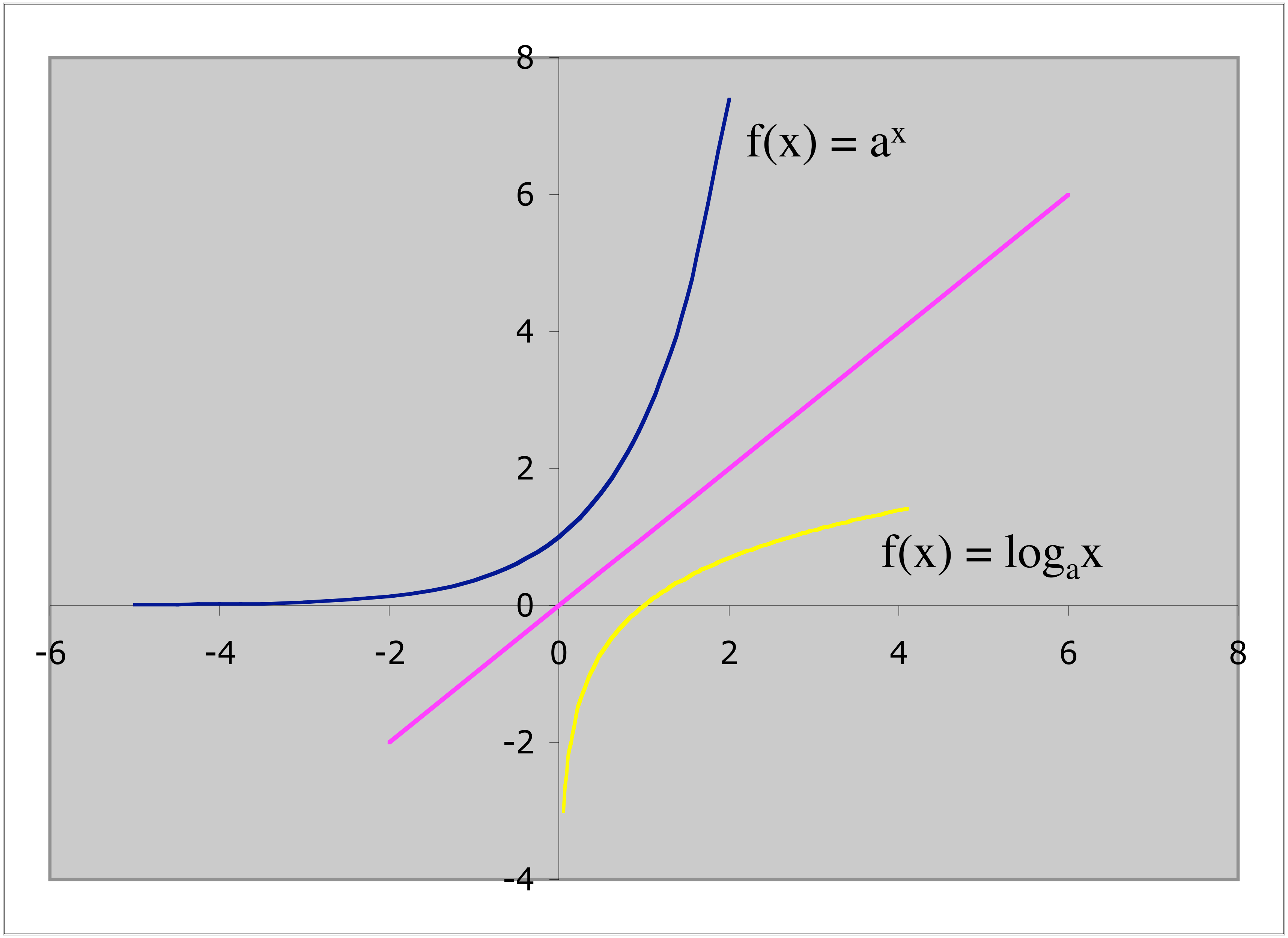
$$\lim_{x \to +\infty} ax = 0$$

- Special case, a = 1



- $f(x) = a^x$ is one-to-one. For every x value there is a unique value of f(x).
- This implies that $f(x) = a^x$ has an inverse.
- $f^{-1}(x) = \log_a x$, logarithm base a of x.





f(x) = a^x
f(x) = log_a x
8
6
4
2
0
-2
-4
-6
-4
-2
0
2
4
6
8

- $\log_a x$ is the power to which *a* must be raised to get x.
- $y = \log_a x$,
- $a^y = x$
- $f(f^{-1}(x)) = a^{\log_a x} = x$, for $x > 0$
- $f^{-1}(f(x)) = \log_a a^x = x$, for all x.
- There are two common forms of the log fn.

  $a = 10$, $\log_{10} x$, commonly written a simply logx

  $a = e = 2.71828\ldots$, $\log_e x = \ln x$, natural log.
- $\log_a x$ does not exist for $x \le 0$.

Laws of logarithms

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(x/y) = \log_a x - \log_a y$
- $\log_a x^k = k \cdot \log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$
- Example 15.7 :

$$2^{3x} = 1.7$$

$$\log_{10} 2^{3x} = \log_{10} 1.7$$

$$3x \log_{10} 2 = \log_{10} 1.7$$

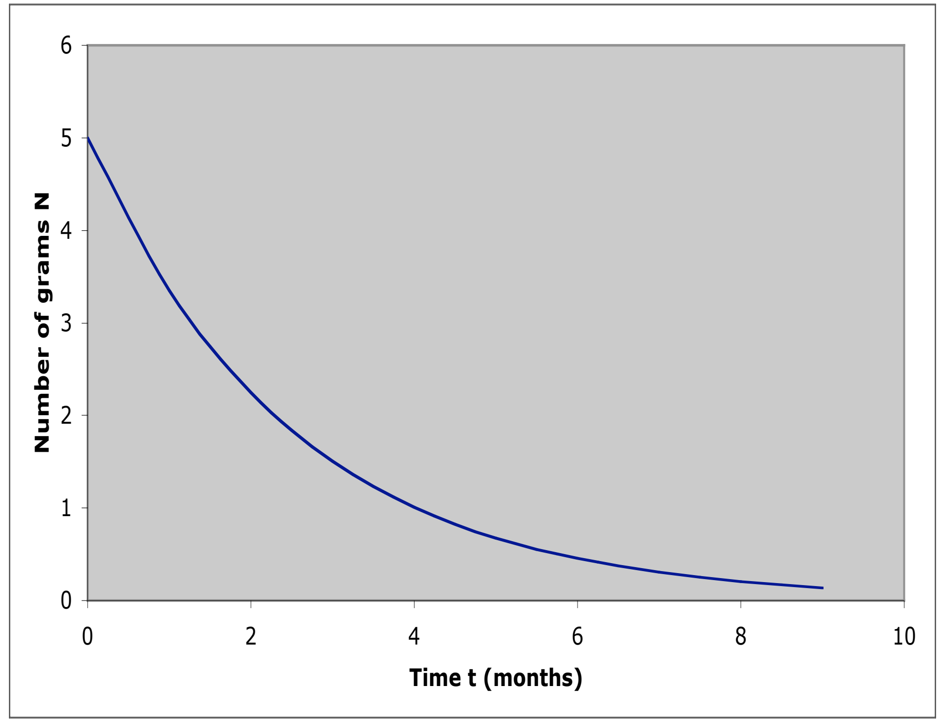
$$x = \frac{\log_{10} 1.7}{3 \log_{10} 2}$$

$$x = \frac{0.2304}{3 * 0.301} = 0.2551$$

- Example 15.8 : Radioactive decay

A radioactive material decays according to the law $N=5e^{-0.4t}$

When does N = 1?
For what value of t does N = 1?



$$1 = 5e^{-0.4t}$$

$$1/5 = e^{-0.4t}$$

$$\ln(0.2) = \ln(e^{-0.4t})$$

$$\ln(0.2) = -0.4t$$

$$t = \ln(0.2)/-0.4$$

$$t = -1.6909/-0.4 = 4.023\ months$$

Trick for computing $\log_a x$ when you calculator doesn't have $\log_a$

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

Example:

$$\log_2 64 = \frac{\ln 64}{\ln 2} = \frac{4.1588...}{0.6931...} = 6$$

- general exponential form:

$f(x) = \square\square^{x}$

$y = \square\square^{x}$

Then

$\ln(y) = \ln \square\square^{x}$

$\ln(y) = \ln\square + \ln\square^{x}$

$\ln(y) = \ln\square + x\ln\square$

Let $b = \ln \square$, and $m = \ln\square$, $Y=\ln y$

$Y \quad = \; b \quad + mx$

which is the equation of a straight line.

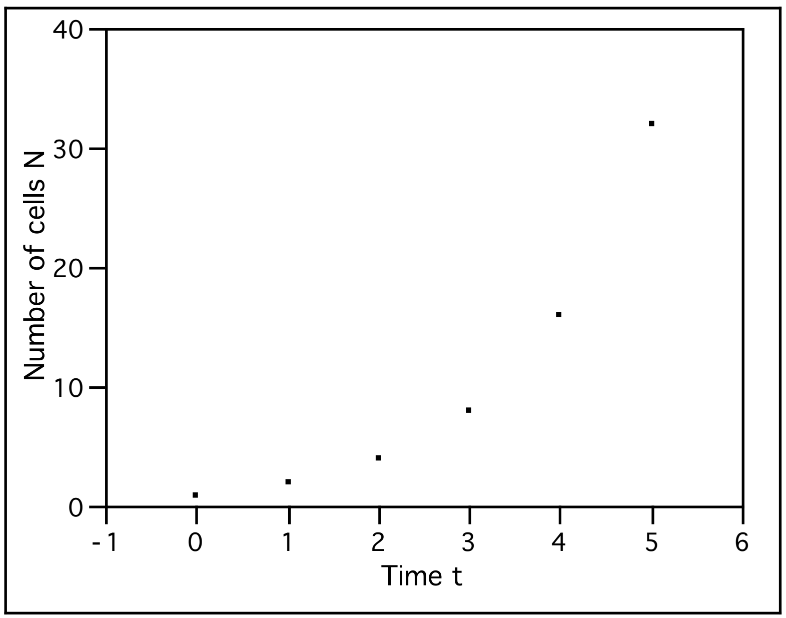
Transform (some) non-linear data so that the transformed data has a linear relationship.

Special exponential form : $f(x) = \square e^{mx}$

Consider the algae growth example again.
How do you know when a relationship is exponential?

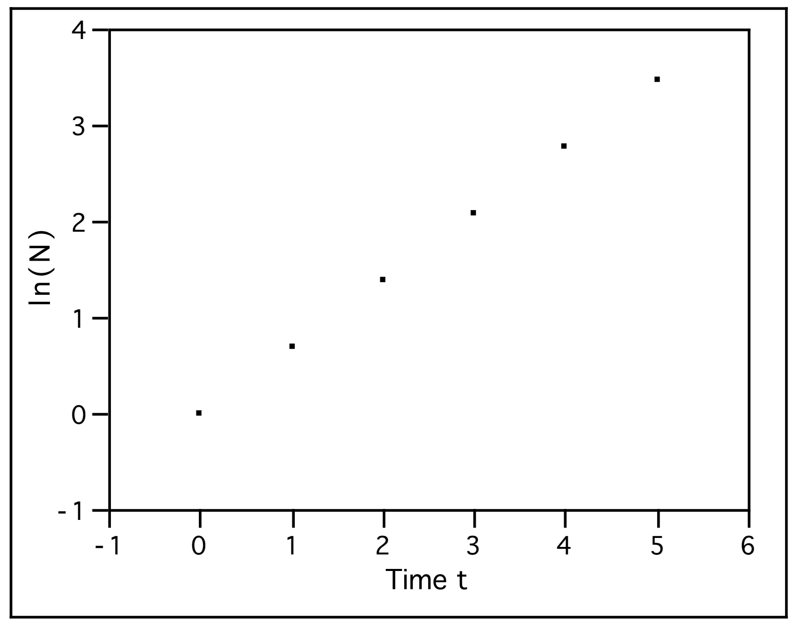
Time t	Number of Cells	ln(N)
0	1	0
1	2	0.69314718
2	4	1.38629436
3	8	2.07944154
4	16	2.77258872
5	32	3.4657359

Regular plot



$$N = \square\square^{t}$$

Semilog plot



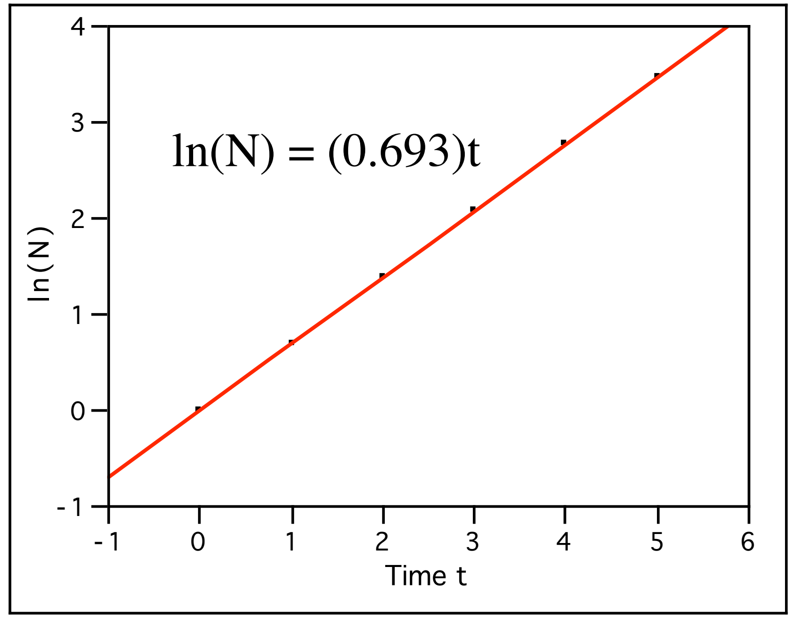
$$\ln(N) = mt+b$$

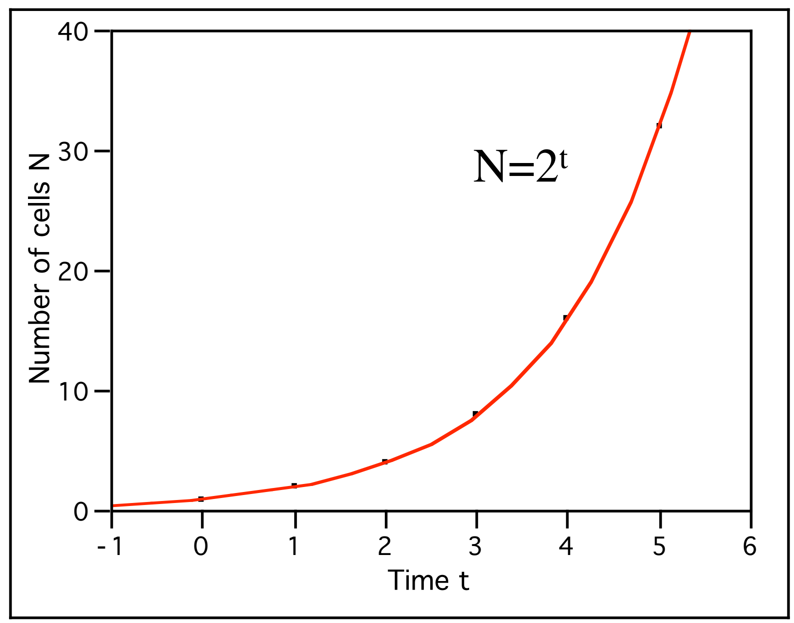
- Fit a line to the transformed data
- Estimate the slope and intercept using the least squares method.
- Y=mx+b
- b~0, m = 0.693..
- Extimate □ and □.

  b = ln□ -> □ = $e^b = e^0 = 1$

  m = ln□ -> □ = $e^m = e^{0.693..} = 2.0$

  $N = 2^t$

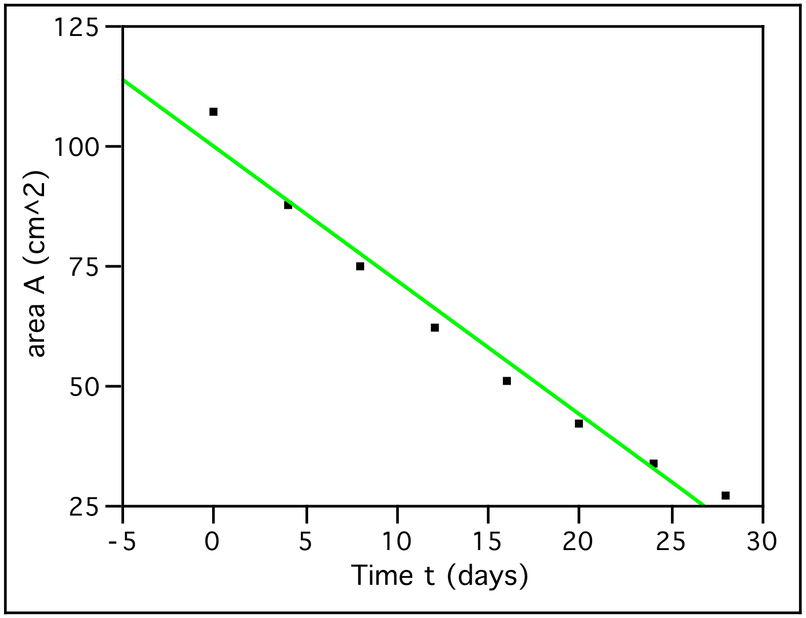




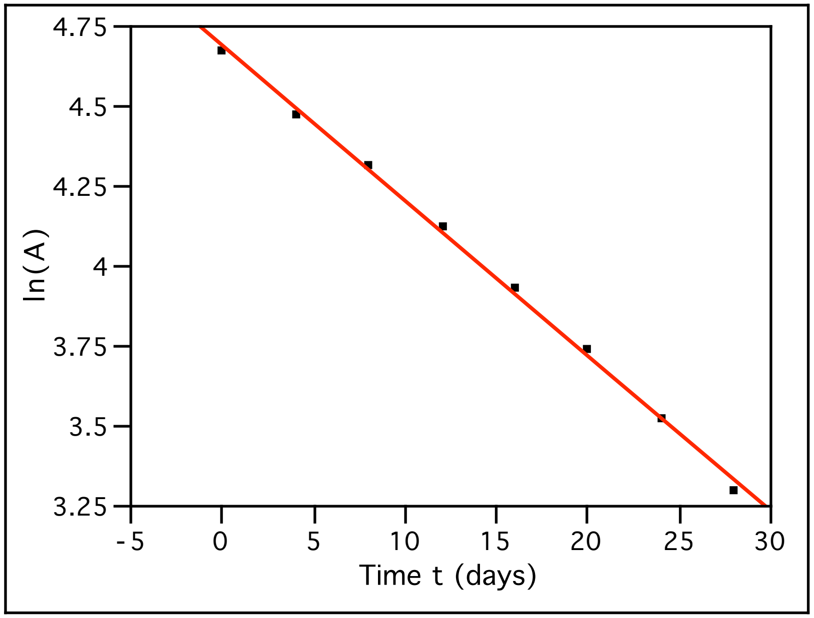
Example 17.9 : Wound healing rate

Time t (days)	Area A (cm^2)	ln(A)
0	107	4.67282883
4	88	4.47733681
8	75	4.31748811
12	62	4.12713439
16	51	3.93182563
20	42	3.73766962
24	34	3.52636052
28	27	3.29583687

Regular plot

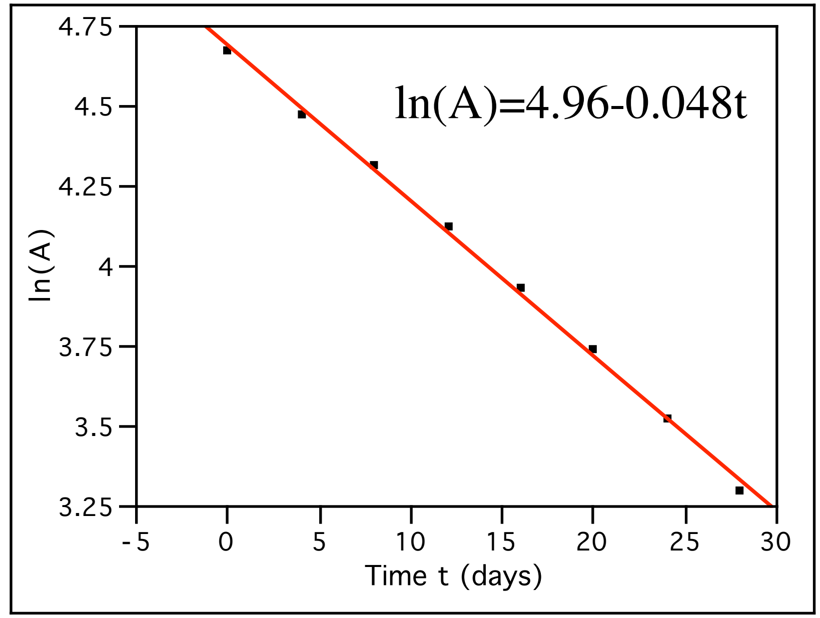


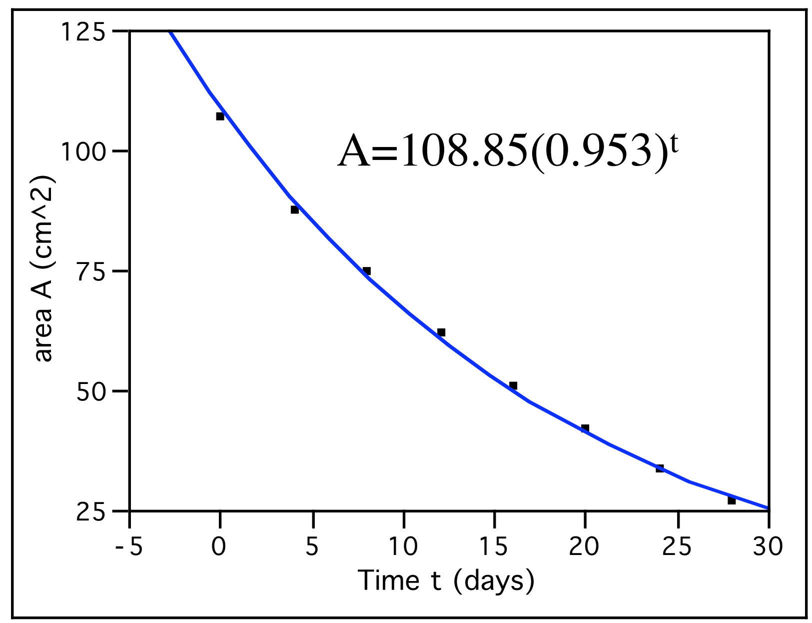
Semilog plot



- How to make a semilog plot?
- Use the semilog(x,y) command in matlab
- Take the log of one column of data and plot the transformed data against the untransformed data

- Estimate slope and intercept using least squares
- $Y = b+mx$
- $m = -0.048$
- $b = 4.69$
- $b = \ln□$ -> $□ = e^{b} = e^{4.69} = 108.85$
- $m = \ln□$ -> $□ = e^{m} = e^{-0.048} = 0.953$
- $A = 108.85(0.953)^{t}$

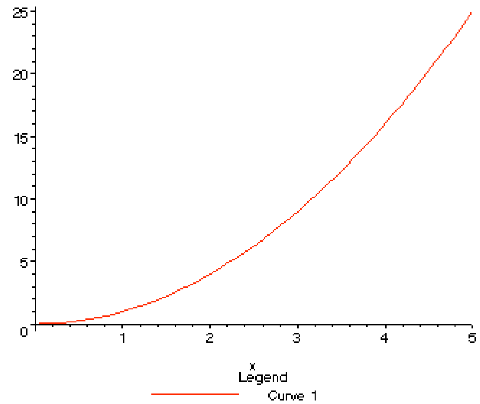




$f(x)=bx^a$

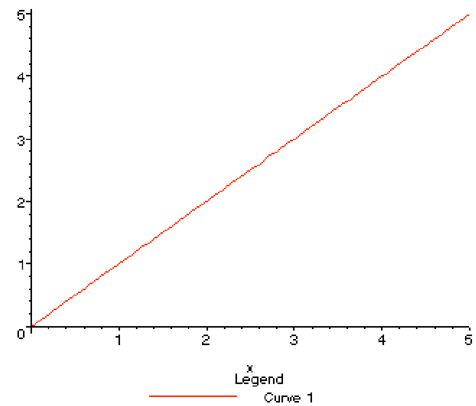
- Allometric relationships
- Descibe the relationship between different aspects of a single organism:
  - Length and volume
  - Surface area and volume
- Typically $x > 0$, since negative quantities don’t have biological meaning.

a > 1



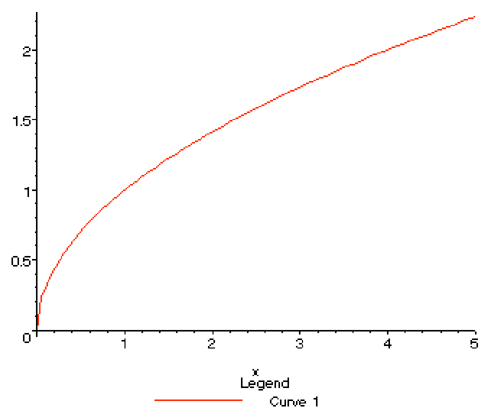
25
20
15
10
5
0
1
2
3
4
5
x
Legend
Curve 1

a = 1

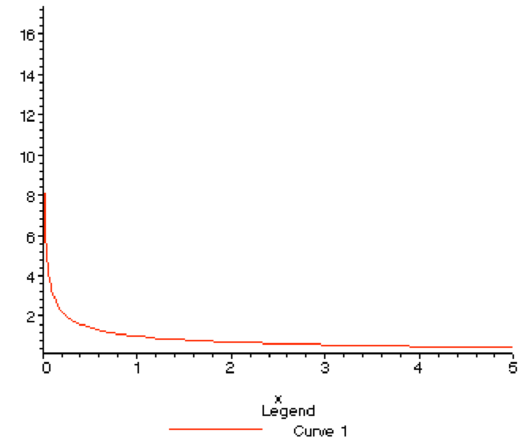


5
4
3
2
1
0
1
2
3
4
5
x
Legend
Curve 1

a < 1

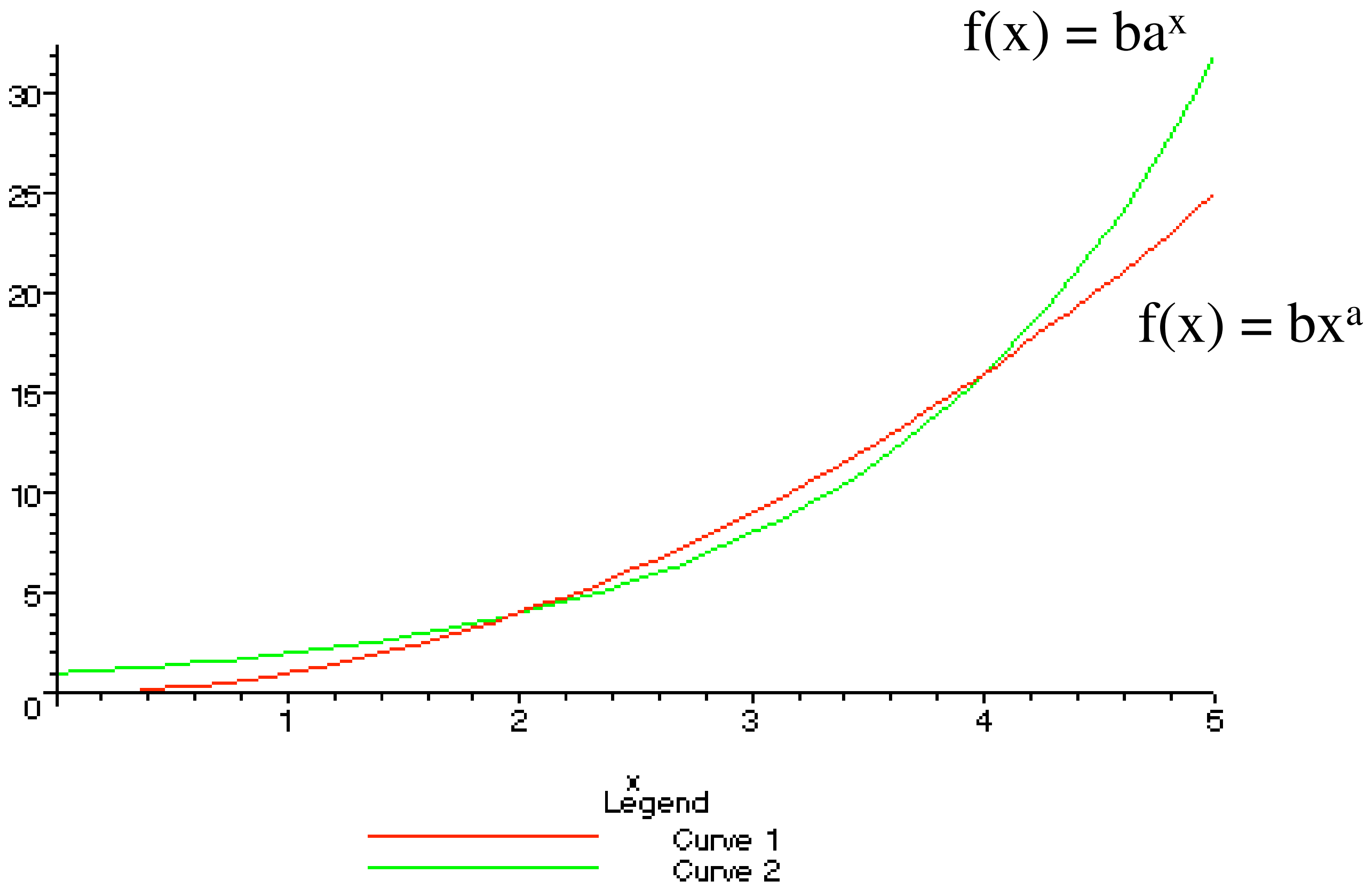


2
1.5
1
0.5
0
1
2
3
4
5
x
Legend
Curve 1



16
14
12
10
8
6
4
2
0
1
2
3
4
5
x
Legend
Curve 1

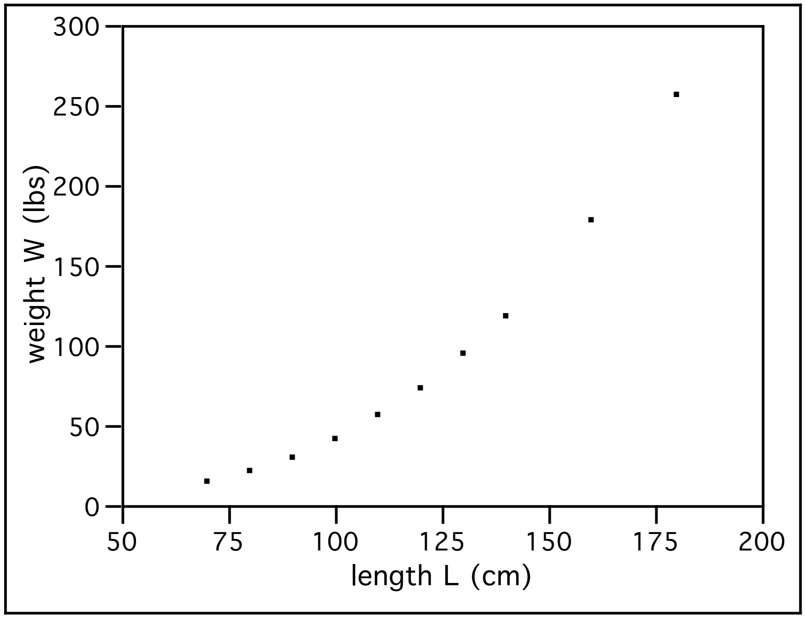
a < 0


f(x) = ba^x
f(x) = bx^a
0
1
2
3
4
5
5
10
15
20
25
30
x
Legend
Curve 1
Curve 2

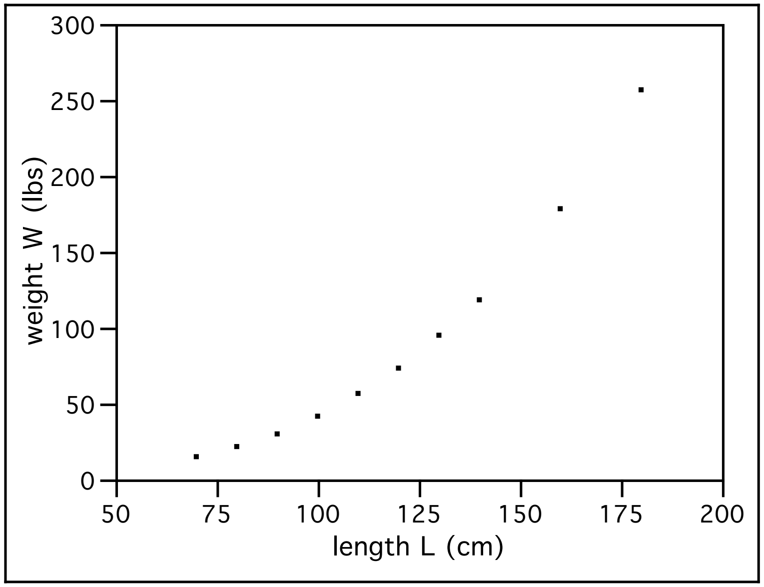
- Example : It has been determined that for any elephant, surface area of the body can be expressed as an allometric function of trunk length.
- For African elephants, a=0.74, and a particular elephant has a suface area of 20 ft^2 and a trunk length of 1 ft.
- What is the surface area of an elephant with a trunk length of 3.3 ft?
- x = trunk length
- y= surface area
- $y = bx^a = bx^{0.74}$
- $20 = b(1)^{0.74}$ $\quad$ $20 = b$
- $y=20x^{0.74}$
- $y=20(3.3)^{0.74}=48.4\ ft^2$

- How do you know when your data has allometric relationship?
- Example 17.10

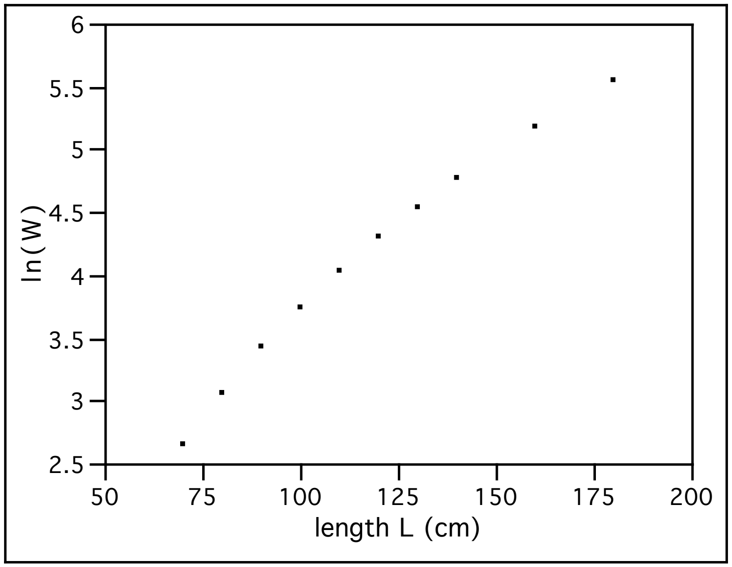
length L (cm)	weight W (lbs)
70	14.3
80	21.5
90	30.8
100	42.5
110	56.8
120	74.1
130	94.7
140	119
160	179
180	256



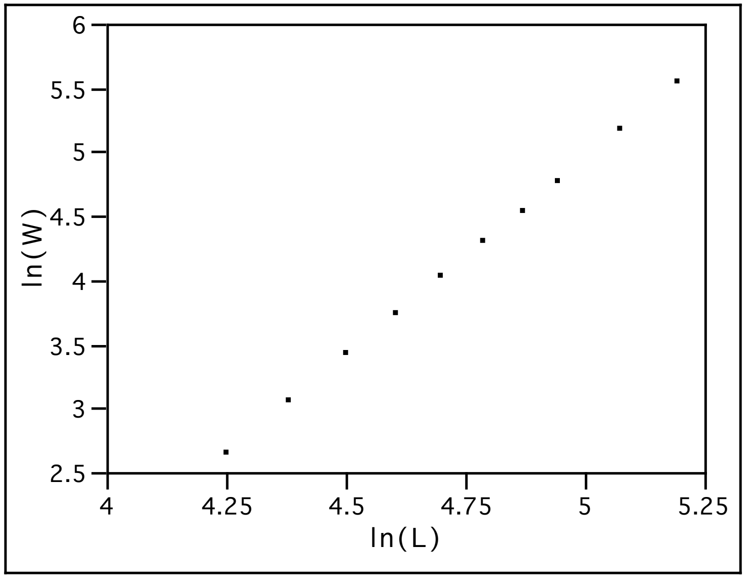
Regular plot



Semilog plot



log-log plot



- How to make a log-log plot
- Use the loglog(x,y) command in matlab
- Take the log of both columns of data and plot the transformed columns.

- $Y=bx^a$
- $\ln(y)=\ln(bx^a)$
- $\ln(y) = \ln(b) + \ln(x^a)$
- $\ln(y) = \ln(b) + a \ln(x)$
- Let

  $Y = \ln(y)$

  $X=\ln(x)$

  $B=\ln(b)$

  Then

  $Y=B + a X$

  Which is the equation for a straight line.