Machine Learning, Nash Equilibria, and Derangements: The Territorial Raider Game

Nina Galanter[1] Dennis Silva, Jr.[2]
Jan Rychtář[3] Jonathan Rowell[4]

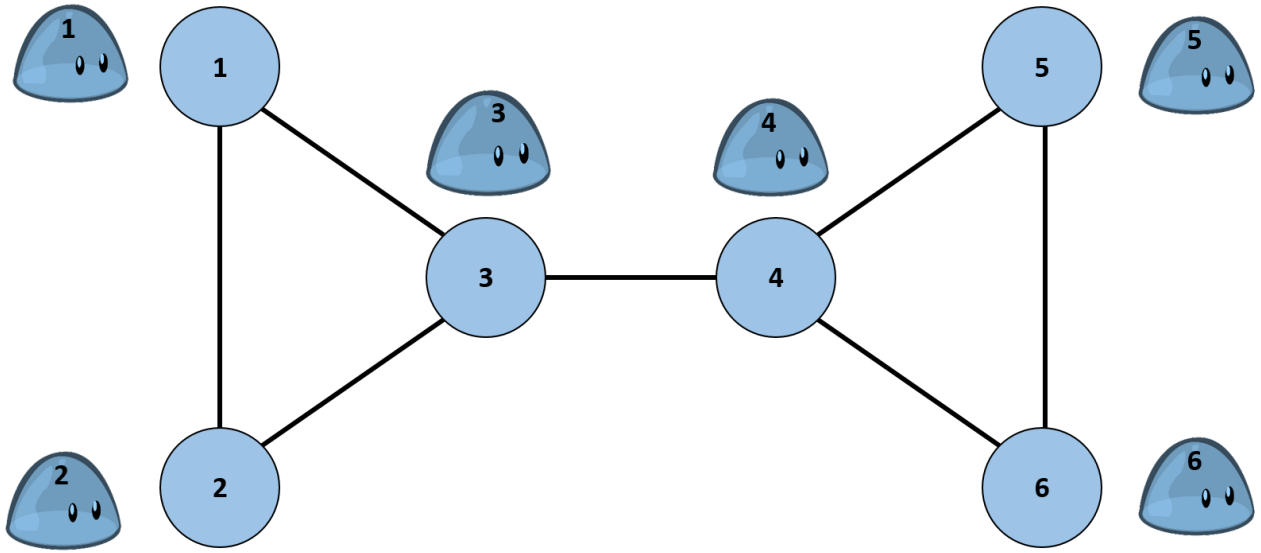
[1]Grinnell College
galanter@grinnell.edu

[2]Worcester Polytechnic Institute
dssilva@wpi.edu

[3]Department of Mathematics and Statistics
University of North Carolina at Greensboro
rychtar@uncg.edu

[4]Department of Mathematics and Statistics
University of North Carolina at Greensboro
jtrowell@uncg.edu

Territorial Raider Model



Rules

1. Each player p_i has a home vertex v_i
2. Each vertex v_i has a given level of resources, R_i
3. Players can choose to raid an adjacent vertex or stay and defend their home
4. If the owner defends, they automatically get some portion $h \in [0, 1]$ of their resources
5. The remaining resources are split between the occupants of a vertex
6. If the owner of a vertex chooses to raid and no one else raids their vertex, they keep their resources
7. The goal is to get as many resources as possible

Definitions

- Strategy Set: A set of all the strategies of the players in a game, i.e. the vertices they move to
- Strict Nash Equilibrium: A strategy set such that if any one player changes their strategy, that player will reduce their payoff

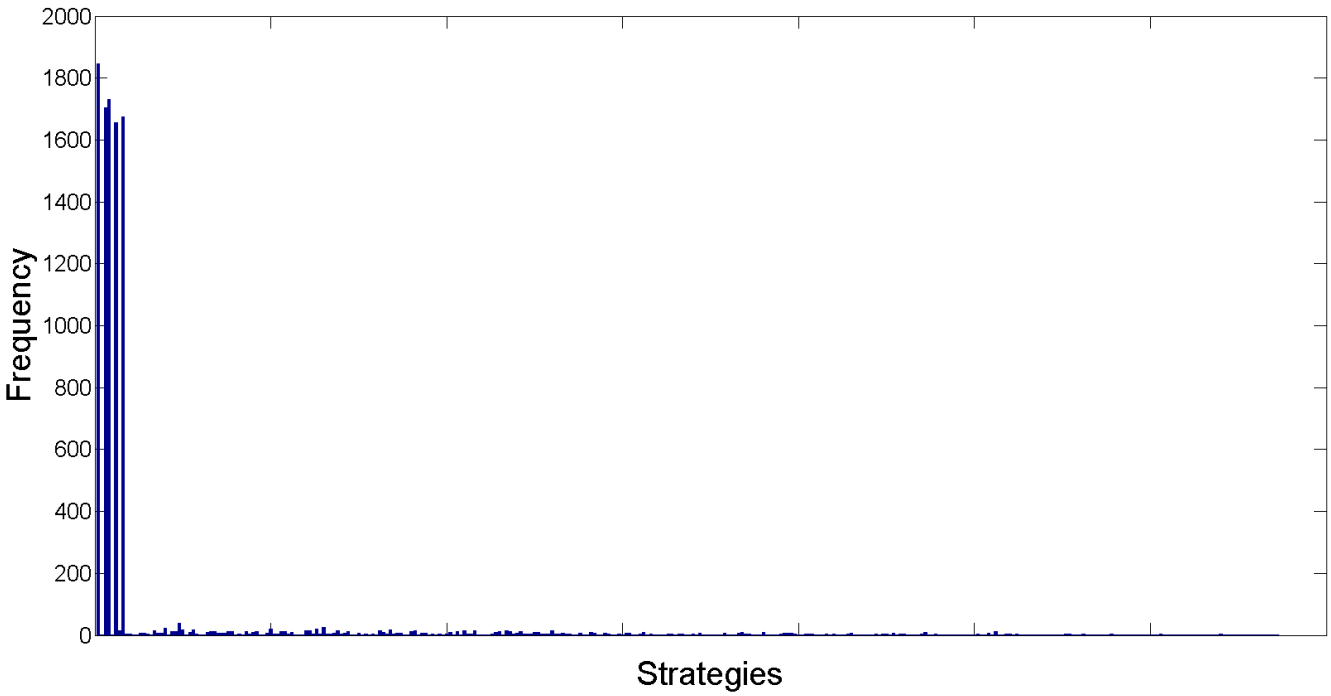
Machine Learning

Exp3 Algorithm Basics(Auer)

1. Starts with a list of possible actions to take
2. Creates a probability distribution of the chance of choosing any one action based on past rewards of these actions
3. Mixes this distribution with the uniform distribution to encourage exploration
4. An action is randomly chosen based on these probabilities
5. Reward is observed and incorporated into a cumulative reward vector
6. Repeat

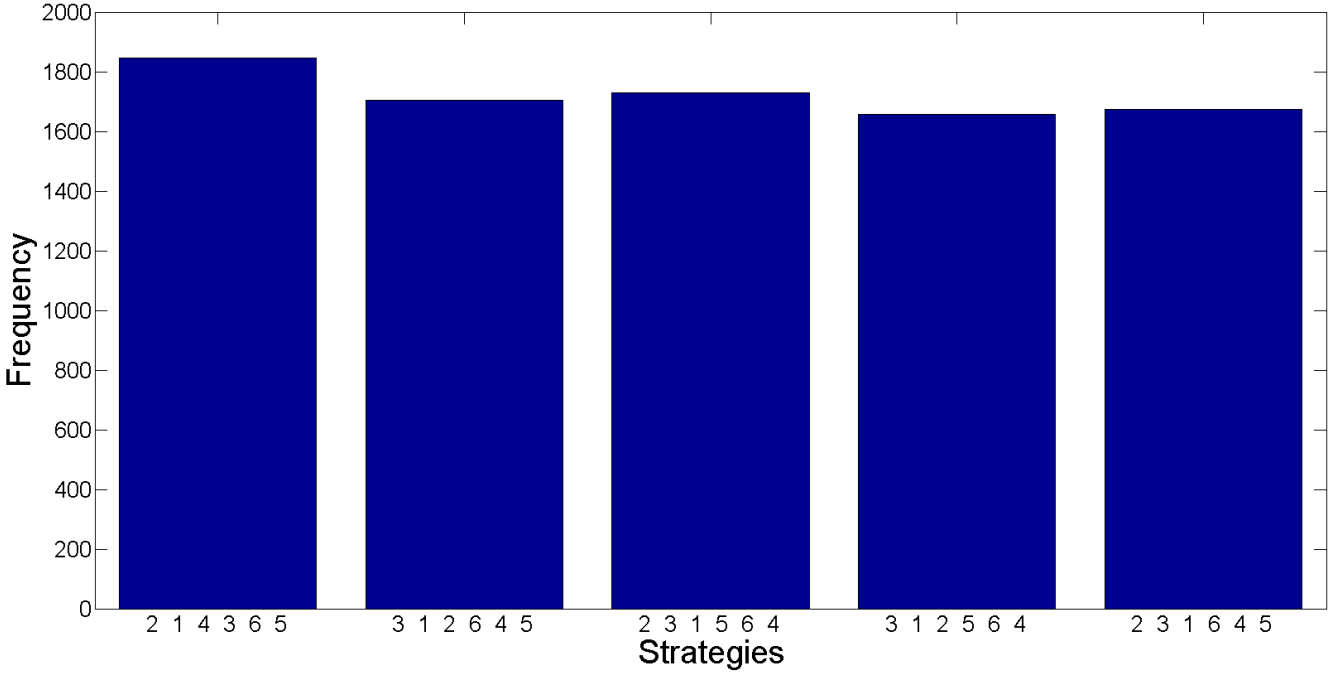
Territorial Raiders using Exp3 Algorithm

Exp3 (Auer) Strategies for Extended Bow Tie Graph
(10,000 Trials)



Territorial Raiders using Exp3 Algorithm

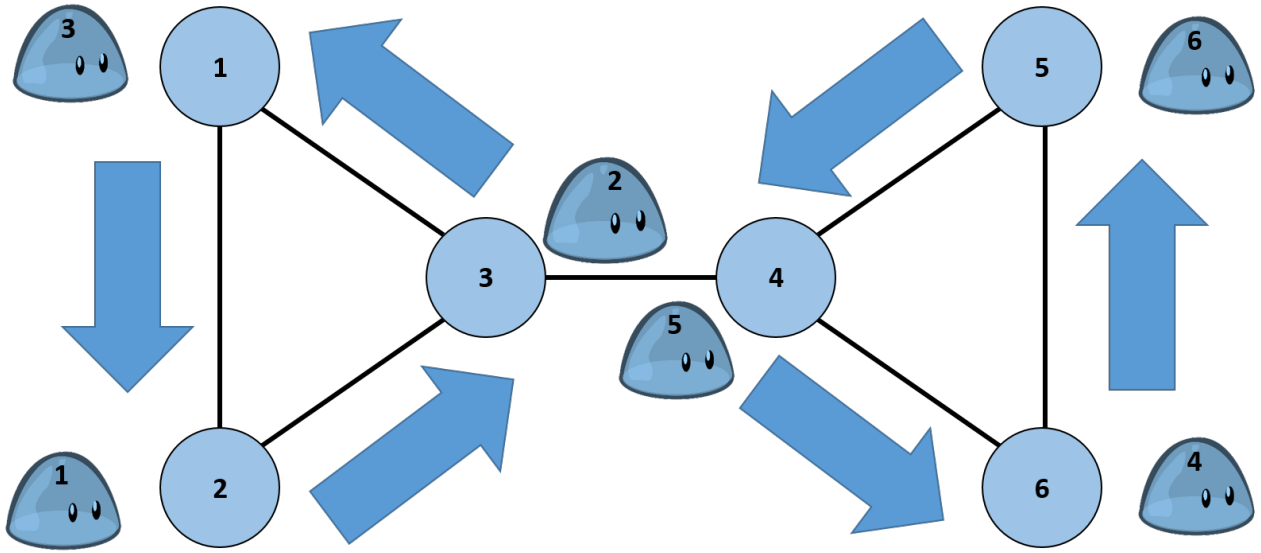
Most Frequent Exp3 Strategies for Extended Bow Tie Graph
(1% Noise Filter)



Derangement

Definition: Let G be a graph with vertex set V. If $f : V \to V$ is an injective function such that for all $v \in V$, $(v, f(v))$ is an edge in G, then f is a derangement of G (Clark).

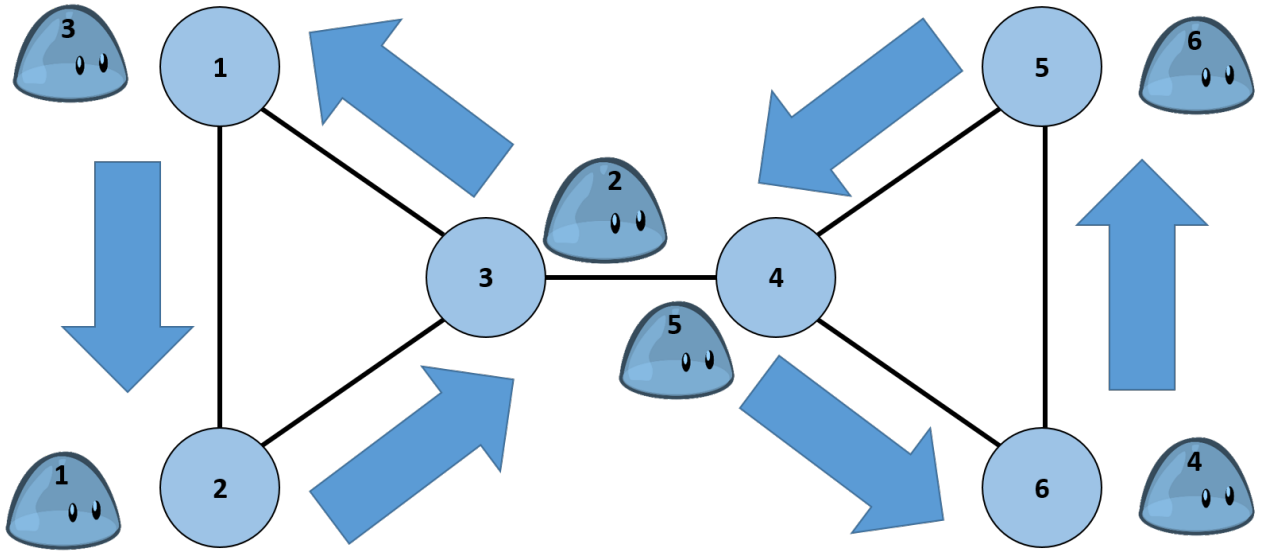
Derangement Example



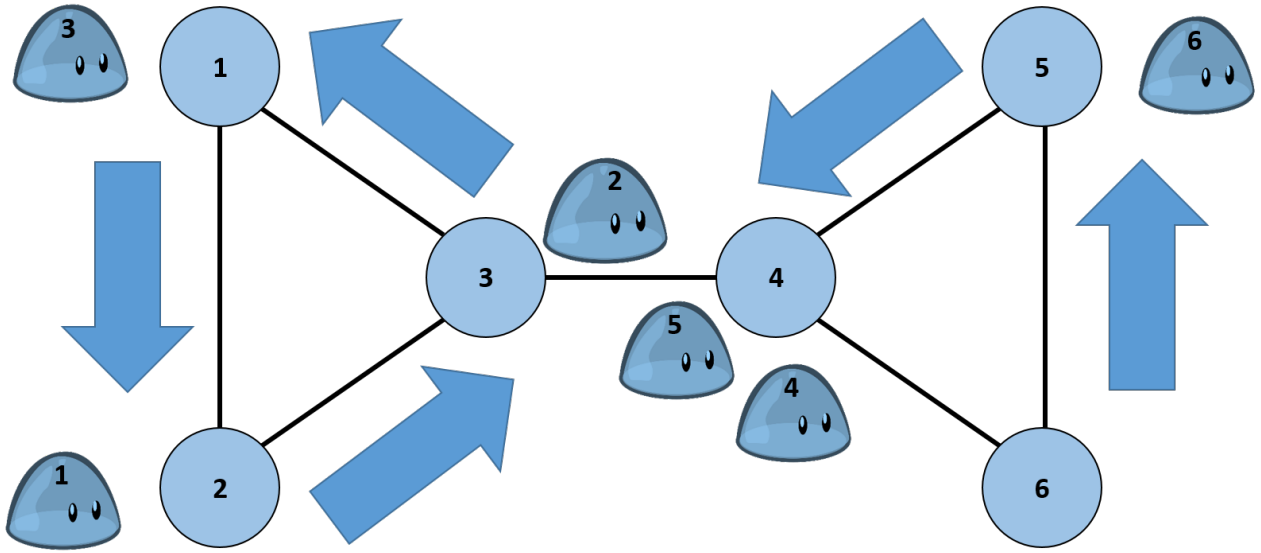
Result

A graph admits a derangement if and only if a Territorial Raider game, with $h \in [0, 1)$, has a strict Nash equilibrium strategy set.

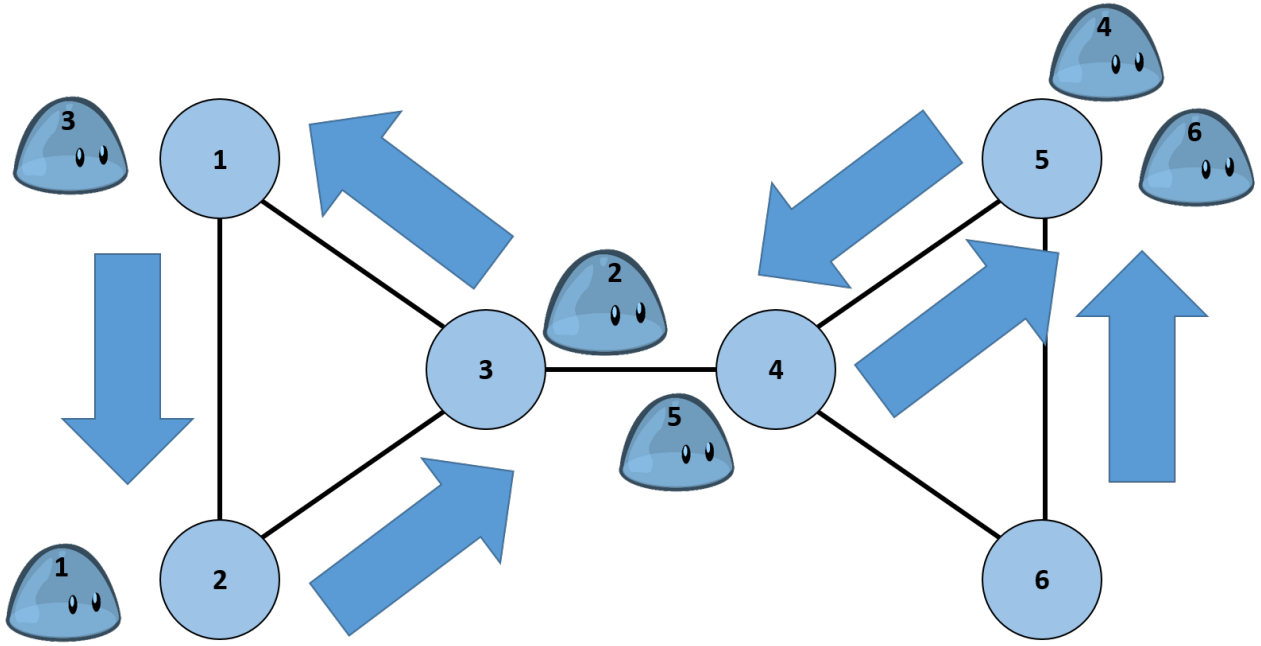
Derangements are Strict Nash Equilibria



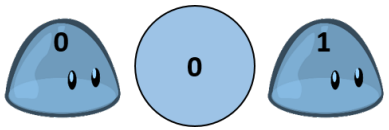
Derangements are Strict Nash Equilibria



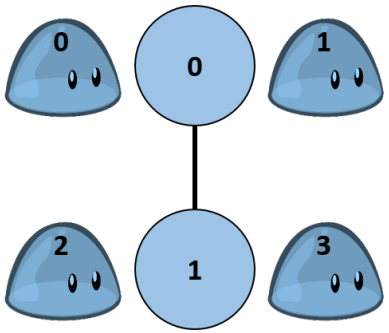
Derangements are Strict Nash Equilibria



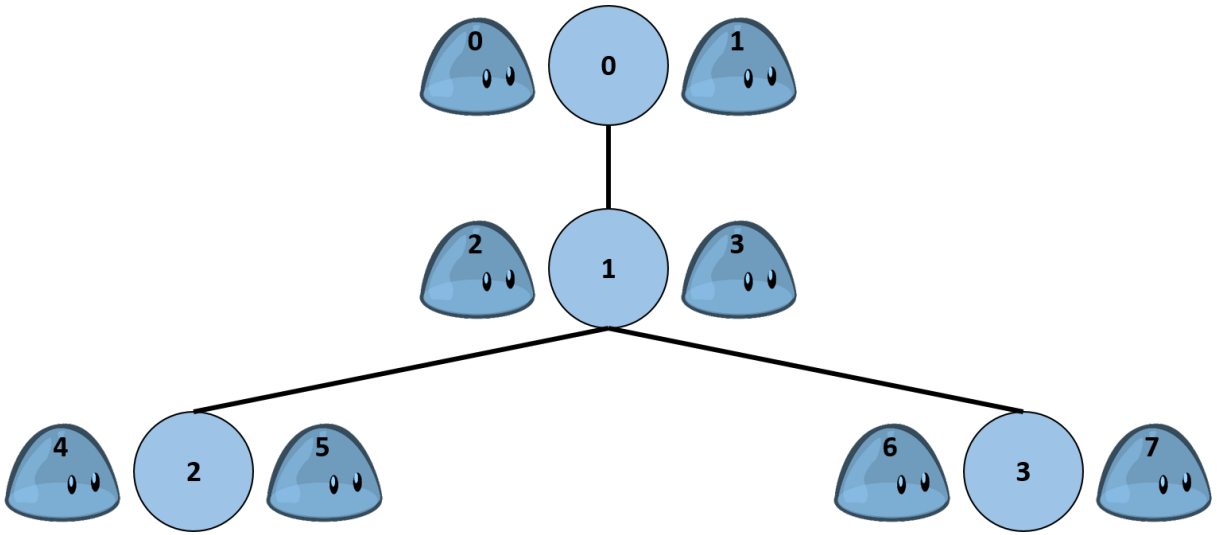
Multiple Players at a Vertex Not Strict Equilibria



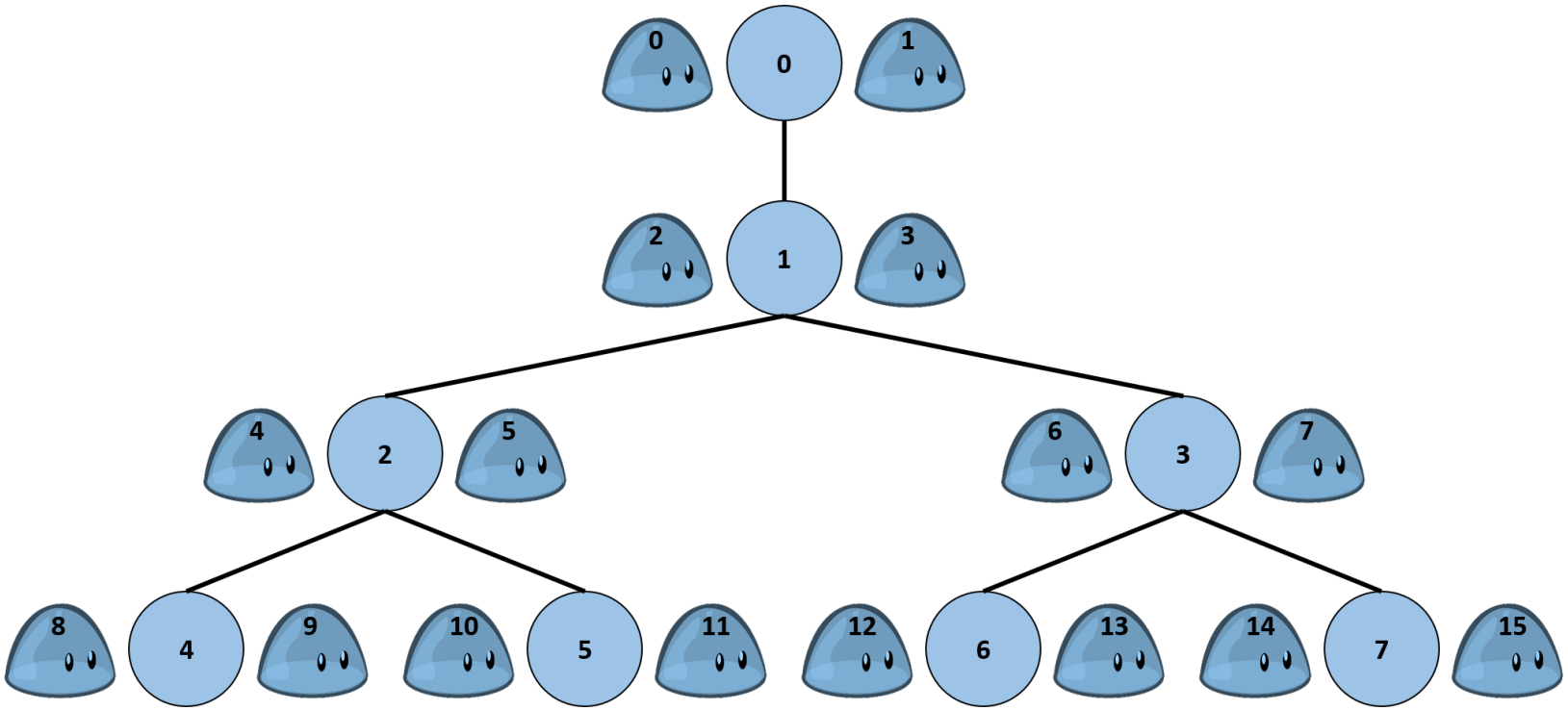
Multiple Players at a Vertex Not Strict Equilibria



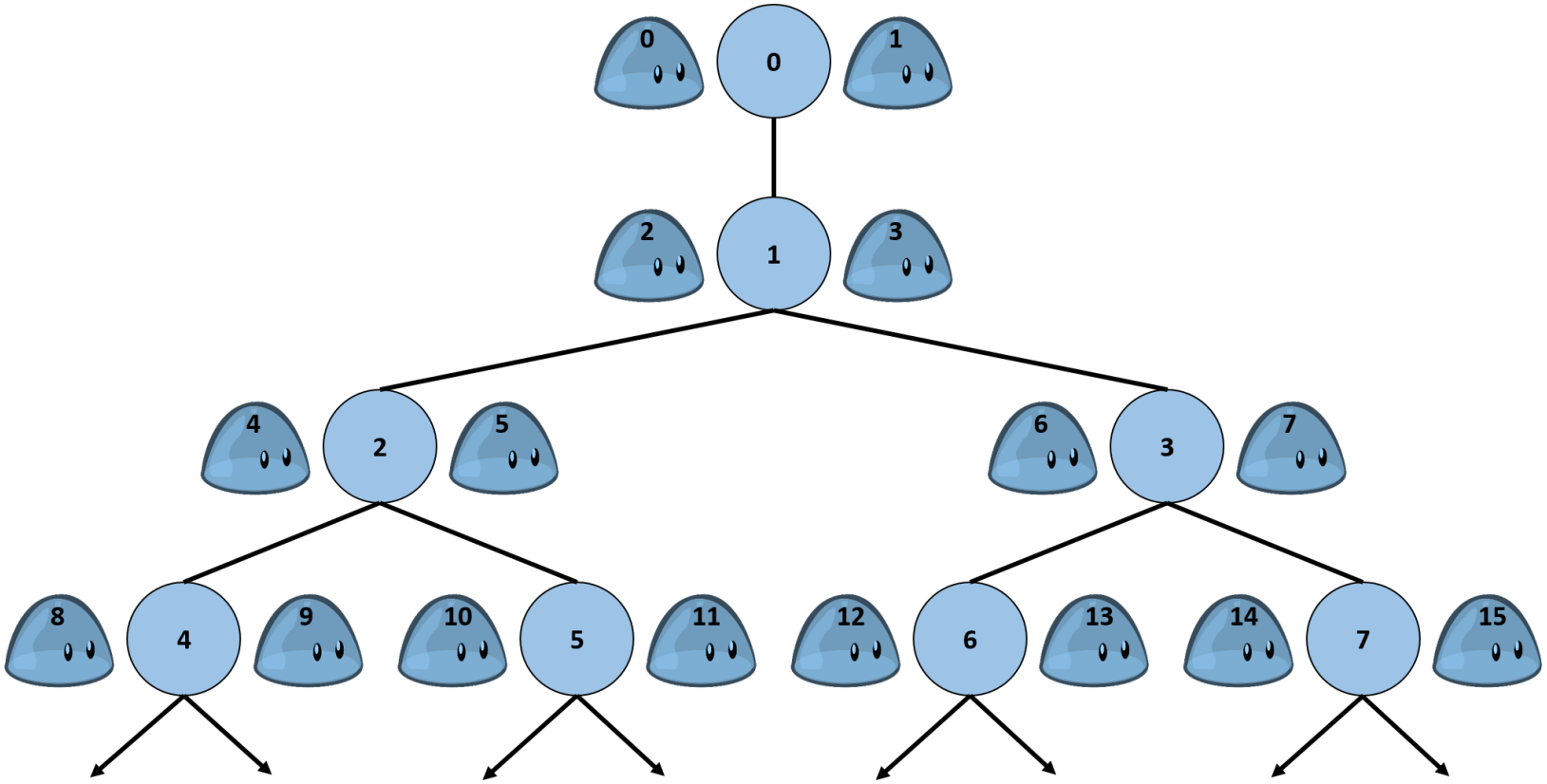
Multiple Players at a Vertex Not Strict Equilibria



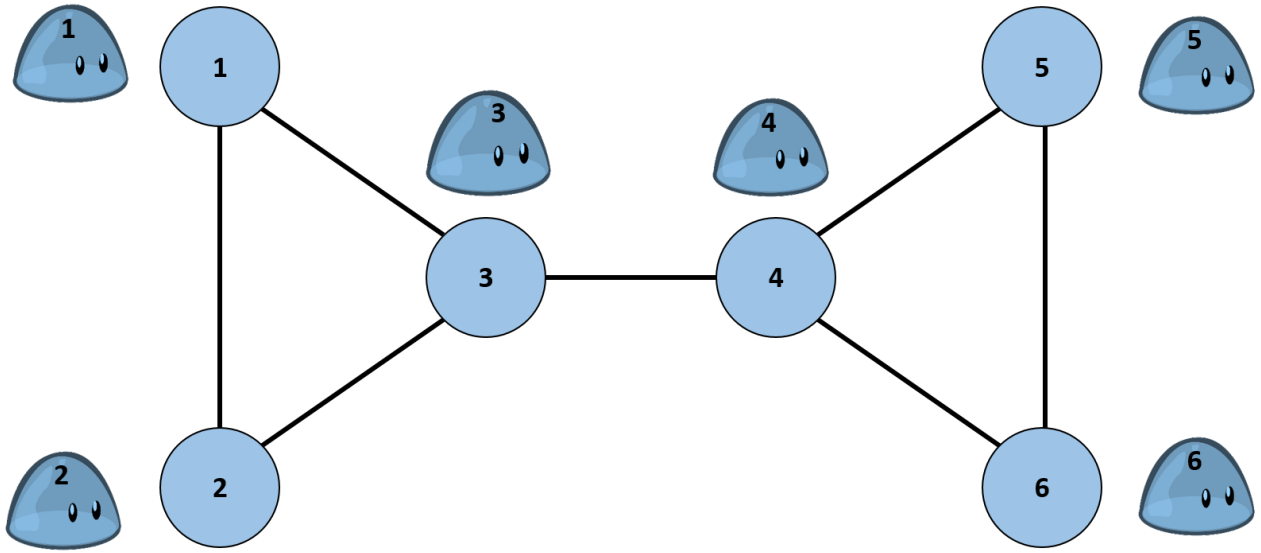
Multiple Players at a Vertex Not Strict Equilibria



Multiple Players at a Vertex Not Strict Equilibria



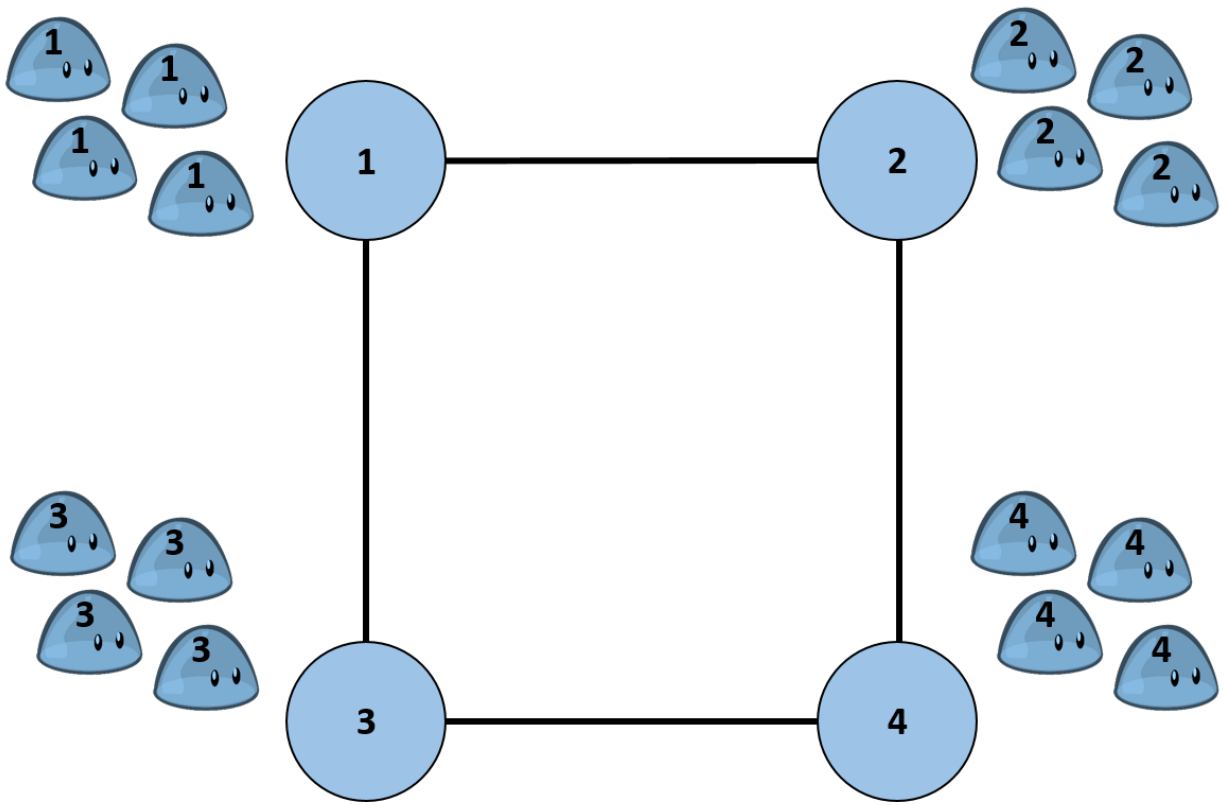
Players at Home Not Strict Equilibria



Significance

- Defined alternate condition for the existence of derangements of a graph
- Found interesting relationship between the structure of the Territorial Raider game, in the form of strict Nash equilibria, and graph structure

Army Territorial Raider Game



Game Rules

1. Each player p_i has a home vertex v_i
2. For discrete game: Each player has U units in their army
3. Each vertex v_i has a given level of resources, R_i
4. A player can both send portions of their army to raid adjacent vertices and keep portions home to defend their own vertex
5. If an owner defends or "protects" with p of their army, they automatically get some portion $H \cdot p$ of their home resources, with $H \in [0, 1]$
6. The remaining resources are split between the occupants of a vertex proportional to the amount of their army that is present
7. If the owner of a vertex chooses to raid and no one else raids their vertex, they keep their resources
8. The goal is to get as many resources as possible

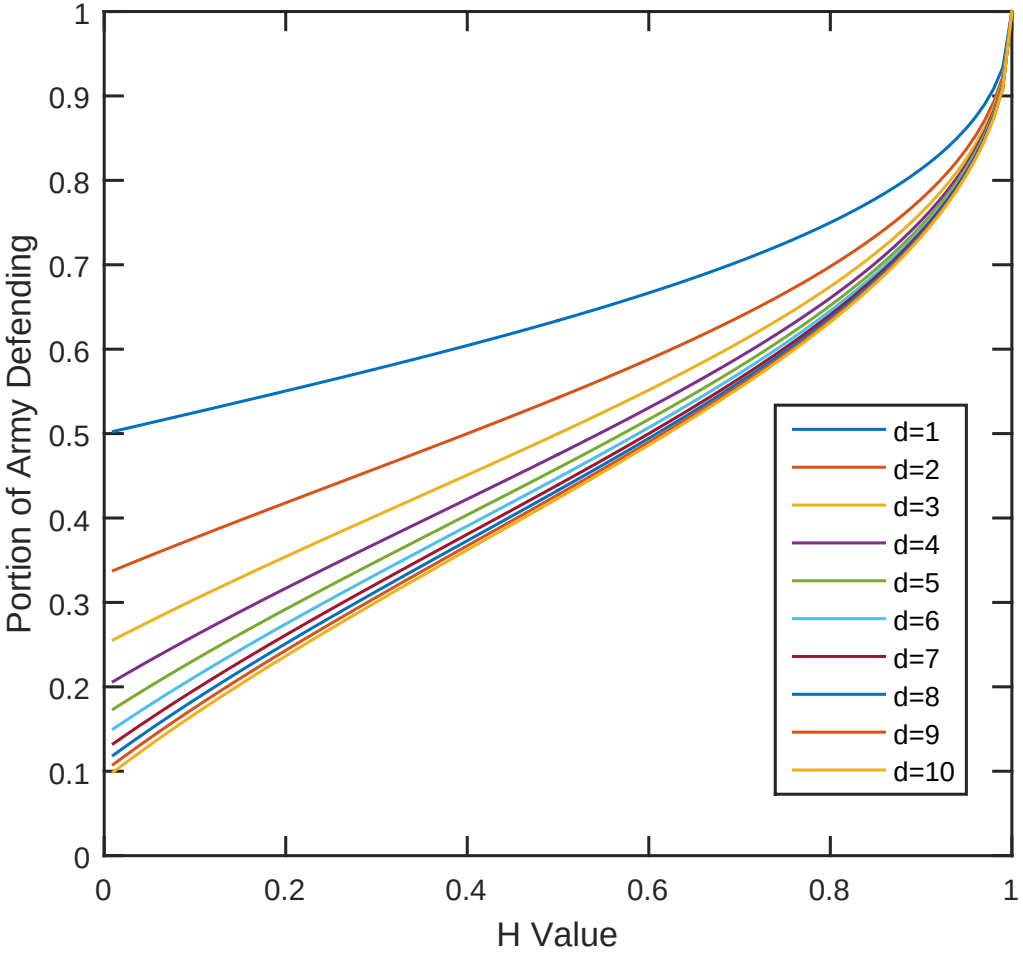
Equilibria Strategies for Regular Graphs: Continuous Armies

$$p^* = \frac{(1+d)(H+1) - \sqrt{(-(H+1)(d+1))^2 - 4H(1+dH)(d+1)}}{2H(d+1)}$$

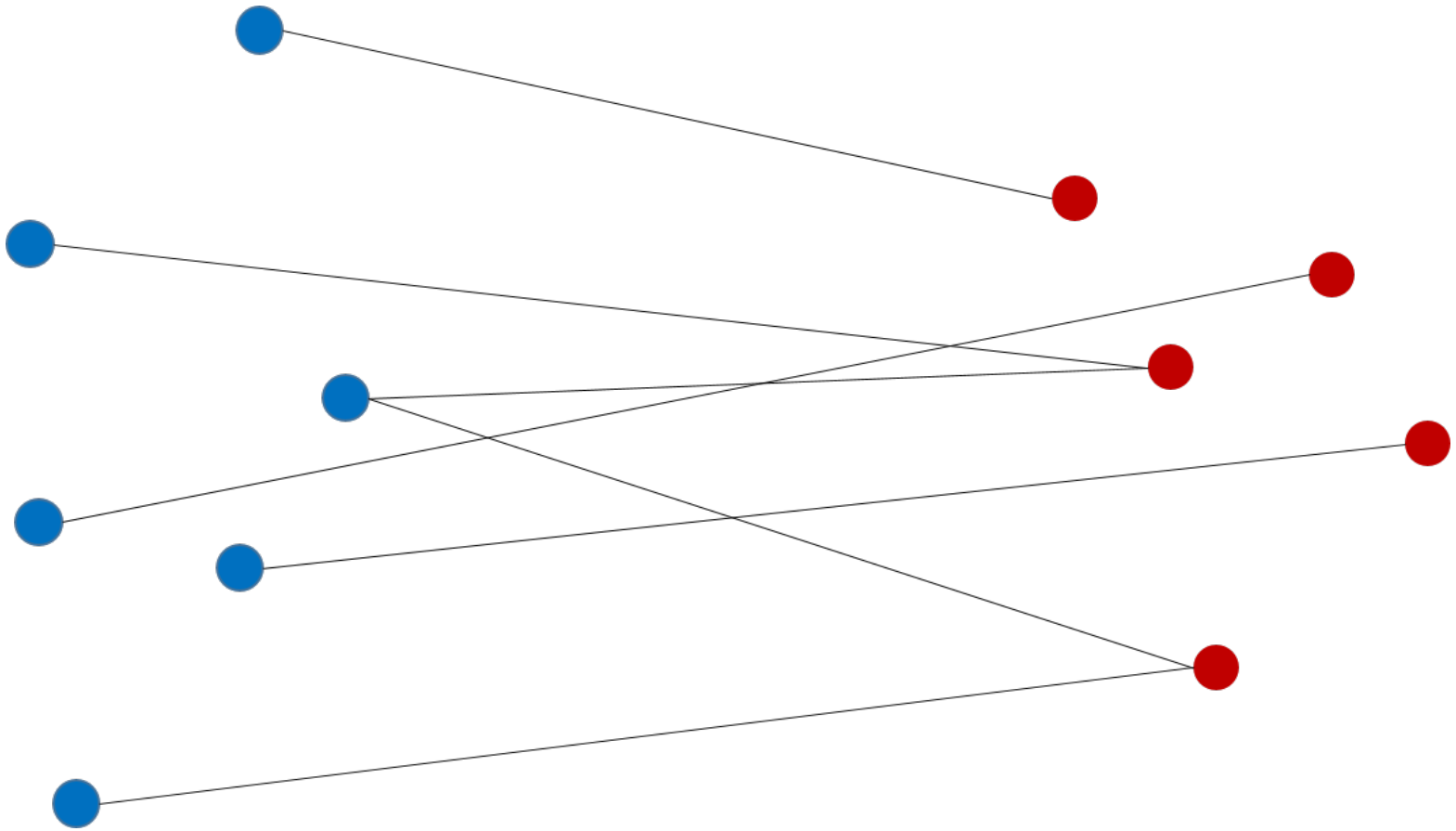
Where

- p^* is the proportion of units to keep home
- d is the degree of the graph (i.e. number of neighbors)

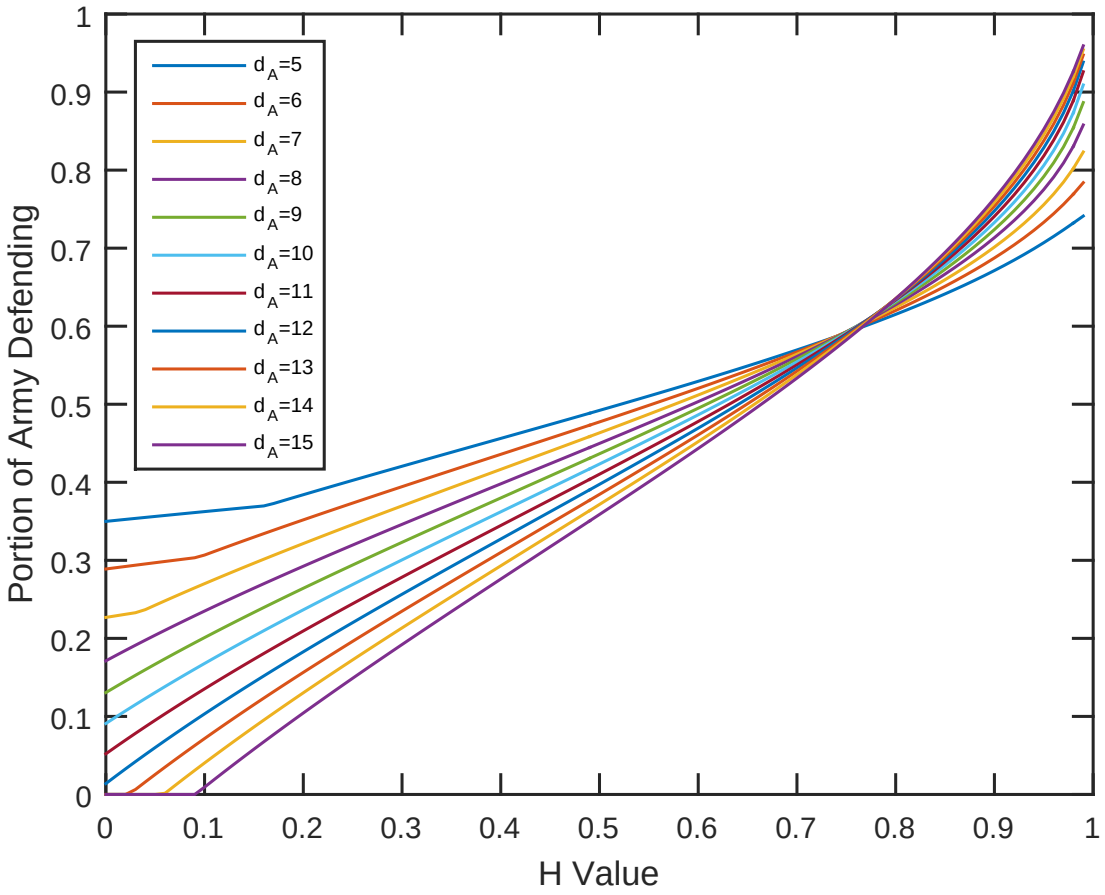
Equilibria Strategies for Regular Graphs: Continuous Armies



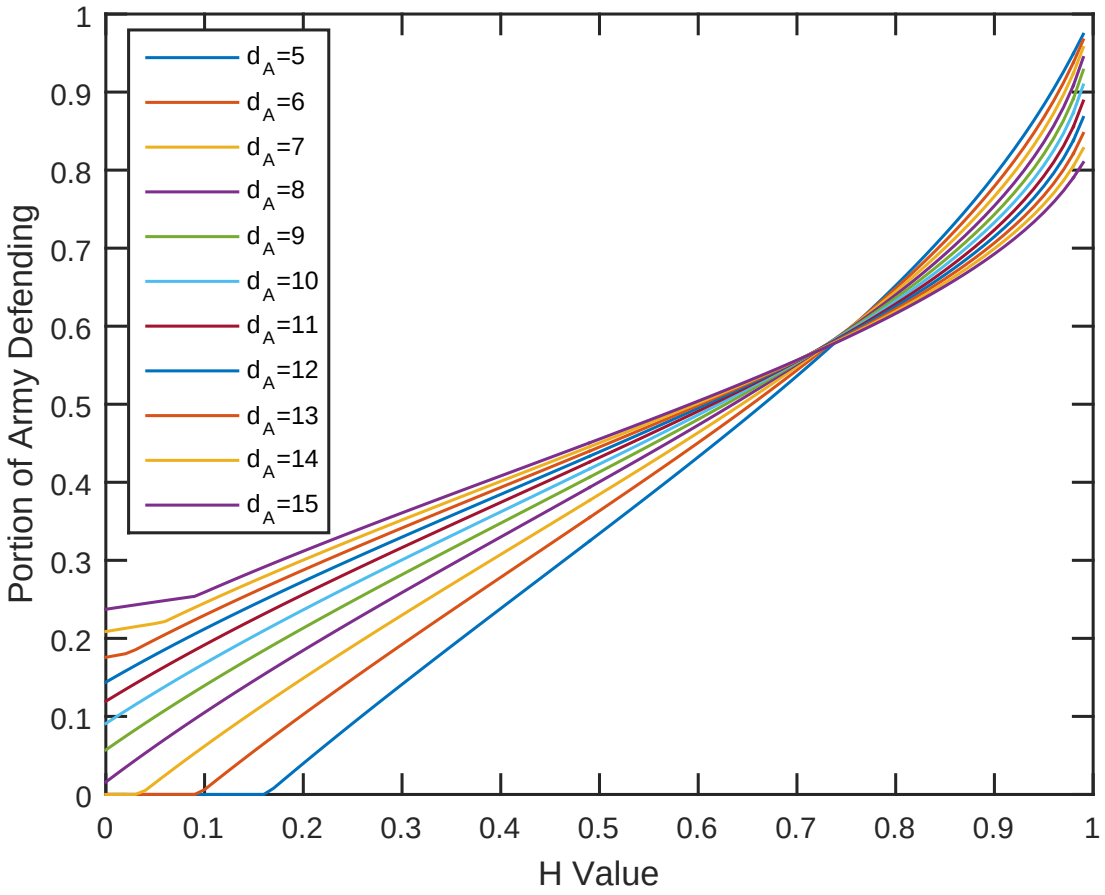
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies



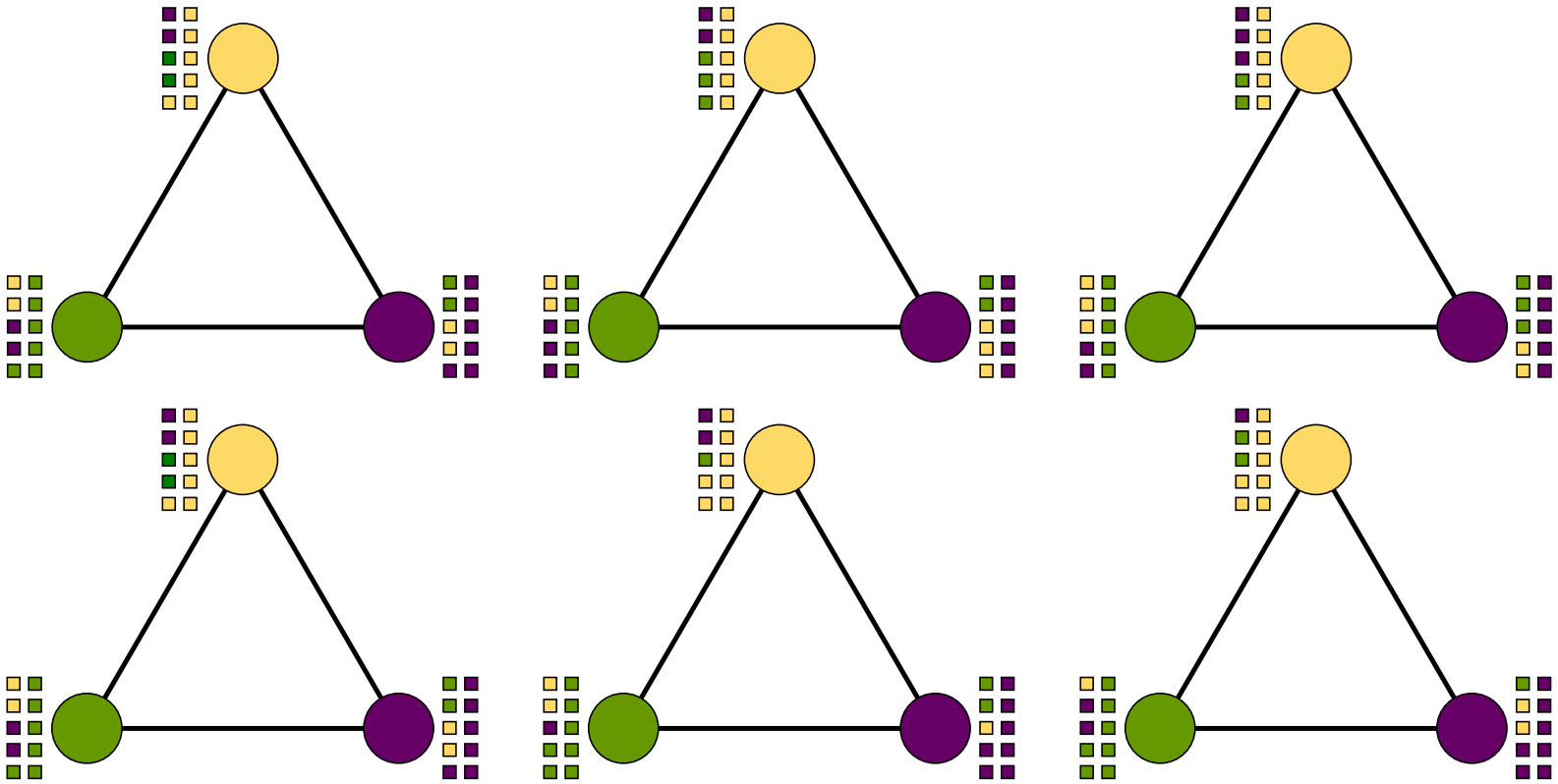
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies - Players in A



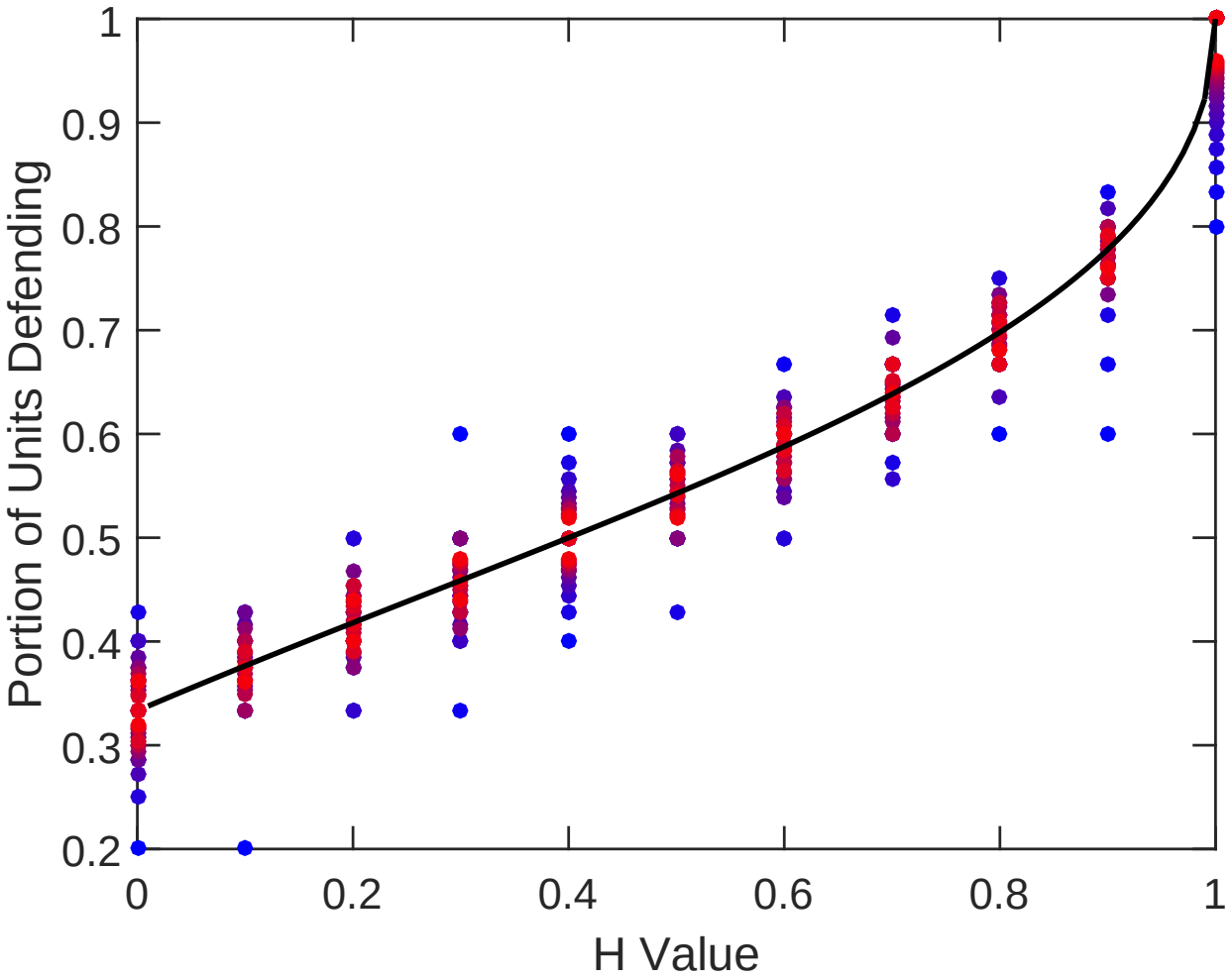
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies - Players in B



Equilibria Strategies for 2-Regular Graphs: Discretely Divisible Armies



Equilibria Strategies for 2-Regular Graphs: Discretely Divisible Armies



Significance

- Extended Territorial Raider model(Broom and Rychtář) to multi-group interactions
- Found that unlike in the individual game, groups at vertices allow for some level of defense to be a strict equilibrium strategy in many cases
- There is a clear effect of graph structure in the form of degree, as well as territory defensibility in the form of H, on strategies

References

- Peter Auer, Nicolo Cesa-Bianchi, Yoav Freund, and Robert E Schapire, *Gambling in a rigged casino: The adversarial multi-armed bandit problem*, Foundations of Computer Science, 1995. Proceedings., 36th Annual Symposium on, IEEE, 1995, pp. 322–331.
- Mark Broom and Jan Rychtář, *A general framework for analysing multiplayer games in networks using territorial interactions as a case study*, Journal of theoretical biology **302** (2012), 70–80.
- Pete L Clark, *Graph derangements*, Open Journal of Discrete Mathematics **3** (2013), no. 04, 183–191.

Acknowledgements

The authors acknowledge the support of Dr. Olav Rueppell, Dr. Clifford Smyth, Dr. Dewey Taylor, Quinn Morris, Catherine Payne, our fellow REU participants, and NSF Grant #1359187.