Machine Learning, Nash Equilibria, and Derangements: The Territorial Raider Game

Nina Galanter\textsuperscript{1}  
Dennis Silva, Jr.\textsuperscript{2}  
Jan Rychtář\textsuperscript{3}  
Jonathan Rowell\textsuperscript{4}

\textsuperscript{1}Grinnell College  
galanter@grinnell.edu

\textsuperscript{2}Worcester Polytechnic Institute  
dssilva@wpi.edu

\textsuperscript{3}Department of Mathematics and Statistics  
University of North Carolina at Greensboro  
rychtar@uncg.edu

\textsuperscript{4}Department of Mathematics and Statistics  
University of North Carolina at Greensboro  
jtrowell@uncg.edu
Territorial Raider Model
Rules

1. Each player $p_i$ has a home vertex $v_i$.
2. Each vertex $v_i$ has a given level of resources, $R_i$.
3. Players can choose to raid an adjacent vertex or stay and defend their home.
4. If the owner defends, they automatically get some portion $h \in [0, 1]$ of their resources.
5. The remaining resources are split between the occupants of a vertex.
6. If the owner of a vertex chooses to raid and no one else raids their vertex, they keep their resources.
7. The goal is to get as many resources as possible.
Definitions

- **Strategy Set**: A set of all the strategies of the players in a game, i.e. the vertices they move to.
- **Strict Nash Equilibrium**: A strategy set such that if any one player changes their strategy, that player will reduce their payoff.
Machine Learning

Exp3 Algorithm Basics (Auer)

1. Starts with a list of possible actions to take
2. Creates a probability distribution of the chance of choosing any one action based on past rewards of these actions
3. Mixes this distribution with the uniform distribution to encourage exploration
4. An action is randomly chosen based on these probabilities
5. Reward is observed and incorporated into a cumulative reward vector
6. Repeat
Territorial Raiders using Exp3 Algorithm

Exp3 (Auer) Strategies for Extended Bow Tie Graph (10,000 Trials)
Territorial Raiders using Exp3 Algorithm

Most Frequent Exp3 Strategies for Extended Bow Tie Graph
(1% Noise Filter)
Derangement

Definition: Let $G$ be a graph with vertex set $V$. If $f : V \to V$ is an injective function such that for all $v \in V$, $(v, f(v))$ is an edge in $G$, then $f$ is a derangement of $G$ (Clark).
Derangement Example

Silva, Galanter, Rowell, Rychtář

Machine Learning for Territorial Raider Model
A graph admits a derangement if and only if a Territorial Raider game, with $h \in [0, 1)$, has a strict Nash equilibrium strategy set.
Derangements are Strict Nash Equilibria
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Multiple Players at a Vertex Not Strict Equilibria
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Multiple Players at a Vertex Not Strict Equilibria
Players at Home Not Strict Equilibria
Significance

- Defined alternate condition for the existence of derangements of a graph
- Found interesting relationship between the structure of the Territorial Raider game, in the form of strict Nash equilibria, and graph structure
Army Territorial Raider Game

Machine Learning for Territorial Raider Model
Game Rules

1. Each player $p_i$ has a home vertex $v_i$
2. For discrete game: Each player has $U$ units in their army
3. Each vertex $v_i$ has a given level of resources, $R_i$
4. A player can both send portions of their army to raid adjacent vertices and keep portions home to defend their own vertex
5. If an owner defends or ”protects” with $p$ of their army, they automatically get some portion $H \cdot p$ of their home resources, with $H \in [0, 1]$
6. The remaining resources are split between the occupants of a vertex proportional to the amount of their army that is present
7. If the owner of a vertex chooses to raid and no one else raids their vertex, they keep their resources
8. The goal is to get as many resources as possible
Equilibria Strategies for Regular Graphs: Continuous Armies

\[ p^* = \frac{(1 + d)(H + 1) - \sqrt{(-(H + 1)(d + 1))^2 - 4H(1 + dH)(d + 1)}}{2H(d + 1)} \]

Where

- \( p^* \) is the proportion of units to keep home
- \( d \) is the degree of the graph (i.e. number of neighbors)
Equilibria Strategies for Regular Graphs: Continuous Armies

![Graph showing the relationship between H Value and Portion of Army Defending for different values of d (1 to 10). The graph indicates that as d increases, the portion of the army defending also increases.]
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies - Players in A
Equilibria Strategies for Regular Bipartite Graphs: Continuous Armies - Players in B
Equilibria Strategies for 2-Regular Graphs: Discretely Divisible Armies
Equilibria Strategies for 2-Regular Graphs: Discretely Divisible Armies

Silva, Galanter, Rowell, Rychtář
Significance

- Extended Territorial Raider model (Broom and Rychtář) to multi-group interactions
- Found that unlike in the individual game, groups at vertices allow for some level of defense to be a strict equilibrium strategy in many cases
- There is a clear effect of graph structure in the form of degree, as well as territory defensibility in the form of $H$, on strategies
References


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