An introduction to stochastic differential equations

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Outline

1. From SRW to BM
2. Stochastic calculus
3. Stochastic differential equations
4. Some examples
5. Continuous state branching process

1. From SRW to BM

Let $\xi_1, \ \xi_2, \ \cdots$ be i.i.d.

$$\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2}$$

Define SRW

$$X_n = \sum_{k=1}^{n} \xi_k.$$

Scaling limit: time step $\frac{1}{n}$ and spatial step $\frac{1}{\sqrt{n}}$

$$B_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k, \qquad t \geq 0.$$

CLT:

$$B_t^n \Rightarrow B_t \sim N(0, t).$$

Definition

(B_t) is a Brownian motion if

- (Normal increments)

$$B_t - B_s \sim N(0, t-s).$$

- (Indep. increments) $B_t - B_s$ indep. of

$$\mathcal{F}_s^B = \sigma(B_u : u \leq s).$$

- (Continuity) $t \mapsto B_t$ is continuous a.s.

Theorem

(B_t) is a Markov process with transition

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

Proof:

$$\begin{aligned}
& \mathbb{P}(B_t \le x | \mathcal{F}_s^B) \\
= \; & \mathbb{P}(B_t - B_s \le x - B_s | \mathcal{F}_s^B) \\
= \; & \int_{-\infty}^{x-B_s} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{|z|^2}{2(t-s)}\right) dz \\
= \; & \int_{-\infty}^{x} p_{t-s}(B_s, y) dy.
\end{aligned}$$

Definition

Let $0 = t_0 < t_1 < \cdots < t_n = t$.

$$[B]_t = \lim_{n \to \infty} \sum_{i=1}^{n} |B_{t_{i+1}} - B_{t_i}|^2$$

is called the quadratic variation process.

Theorem

$$[B]_t = t, \qquad a.s.$$

Proof: Denote

$$T_n = \sum_{i=1}^{n} |B_{t_{i+1}} - B_{t_i}|^2.$$

Then,

$$\mathbb{E}(T_n) = \sum_{i=1}^{n} (t_{i+1} - t_i) = t$$

and hence,

$$\begin{aligned} \mathbb{E}(|T_n - t|^2) &= Var(T_n) \\ &= \sum_{i=1}^{n} Var\left(|B_{t_{i+1}} - B_{t_i}|^2\right) \\ &= \sum_{i=1}^{n} 2(t_{i+1} - t_i)^2 = \frac{2t^2}{n} \to 0. \end{aligned}$$

Corollary

$t \mapsto B_t$ is nowhere differentiable a.s.

2. Stochastic calculus

Let $0 = t_0 < t_1 < \cdots < t_n = t$.

Recall Riemann integral:

$$\int_0^t f_s ds = \lim_{n\to\infty} \sum_{i=1}^{n-1} f_{s_i}(t_{i+1} - t_i)$$

where $s_i \in [t_i, t_{i+1}]$.

Definition

$$\int_0^t f_s dB_s = \lim_{n\to\infty} \sum_{i=1}^{n-1} f_{t_i}(B_{t_{i+1}} - B_{t_i})$$

is called the Itô integral of f wrt B.

Theorem

$$\int_0^t (\alpha f_s + \beta g_s) dB_s = \alpha \int_0^t f_s dB_s + \beta \int_0^t g_s dB_s,$$

$$\mathbb{E} \int_0^t f_s dB_s = 0$$

and

$$\mathbb{E} \left| \int_0^t f_s dB_s \right|^2 = \int_0^t \mathbb{E} f_s^2 ds.$$

Proof:

$$
\begin{aligned}
& \mathbb{E}\left|\int_0^t f_s dB_s\right|^2 \\
= & \lim_{n\to\infty} Var\left(\sum_{i=1}^{n-1} f_{t_i}(B_{t_{i+1}} - B_{t_i})\right) \\
= & \lim_{n\to\infty} \sum_{i=1}^{n-1} Var\left(f_{t_i}(B_{t_{i+1}} - B_{t_i})\right) \\
= & \lim_{n\to\infty} \sum_{i=1}^{n-1} \mathbb{E}\left(|f_{t_i}|^2 |B_{t_{i+1}} - B_{t_i}|^2\right) \\
= & \lim_{n\to\infty} \sum_{i=1}^{n-1} \mathbb{E}|f_{t_i}|^2 (t_{i+1} - t_i).
\end{aligned}
$$

Recall "Chain rule" in calculus: If g_t is differentiable and $y_t = f(g_t)$, then

$$\frac{dy_t}{dt} = f'(g_t)\frac{dg_t}{dt}.$$

Namely,

$$dy_t = f'(g_t)dg_t.$$

So

$$f(g_t) = f(g_0) + \int_0^t f'(g_s)dg_s.$$

Itô's formula

Theorem

$\forall f \in C^2(\mathbb{R})$, we have

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

In differential form:

$$\begin{aligned} df(B_t) &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 \\ &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt. \end{aligned}$$

Proof: Let $0 = t_0 < t_1 < \cdots < t_n = t$.

$$\begin{aligned} & f(B_t) - f(0) \\ = & \sum_{i=0}^{n-1} (f(B_{t_{i+1}}) - f(B_{t_i})) \\ = & \sum_{i=0}^{n-1} (f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2} f''(B_{s_i})(B_{t_{i+1}} - B_{t_i})^2) \\ = & I_1^n + I_2^n, \end{aligned}$$

where $s_i \in [t_i, t_{i+1}]$.

$$I_1^n \to \int_0^t f'(B_s) dB_s.$$

To estimate I_2^n, we note that

$$\begin{aligned}
& \mathbb{E}\left|\sum_{i=0}^{n-1} f''(B_{s_i})\left((B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\right)\right|^2 \\
\approx & \sum_{i=0}^{n-1} \mathbb{E}|f''(B_{t_i})^2 \mathbb{E}\left|(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\right|^2 \\
\leq & \; K \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \\
= & \; K t^2 n^{-1} \to 0.
\end{aligned}$$

Hence,

$$I_2^n \approx \frac{1}{2} f''(B_{s_i})(t_{i+1} - t_i) \to \frac{1}{2}\int_0^t f''(B_s)ds.$$

Example Let $X_t = \exp(B_t)$. Find dX_t.
Solution:

$$dX_t = \exp(B_t)dB_t + \frac{1}{2}\exp(B_t)dt$$

So

$$X_t = 1 + \int_0^t e^{B_s}dB_s + \frac{1}{2}\int_0^t e^{B_s}ds.$$

Definition

$$\int_0^t f_s \partial B_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{f_{t_i} + f_{t_{i+1}}}{2} (B_{t_{i+1}} - B_{t_i})$$

is the Stratanovich integral of f wrt B.

Theorem

$$\int_0^t f_s \partial B_s = \int_0^t f_s dB_s + \frac{1}{2}[f, B]_t$$

where $[f, B]_t$ *is the quadratic covariation process.*

Proof:

$$\begin{aligned}
\int_0^t f_s \partial B_s &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2f_{t_i} + f_{t_{i+1}} - f_{t_i}}{2}(B_{t_{i+1}} - B_{t_i}) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} f_{t_i}(B_{t_{i+1}} - B_{t_i}) \\
&\quad + \frac{1}{2} \lim_{n\to\infty} \sum_{i=0}^{n-1} (f_{t_{i+1}} - f_{t_i})(B_{t_{i+1}} - B_{t_i}) \\
&= \int_0^t f_s dB_s + \frac{1}{2}[f, B]_t.
\end{aligned}$$

Corollary

$$\int_0^t B_s \partial B_s = \int_0^t B_s dB_s + \frac{1}{2}t$$

Itô formula for Stratanovich integral

Theorem

$$\partial f(B_t) = f'(B_t)\partial B_t,$$

namely,

$$f(B_t) = f(0) + \int_0^t f'(B_s)\partial B_s.$$

3. Stochastic differential equations

ODE

$$\dot{X}_t = b(X_t)$$

ODE with white noise perturbation

$$\dot{X}_t = b(X_t) + n_t$$

Model WN by derivative of BM: $n_t = \dot{W}_t$.
Also assume the magnitude of the perturbation depend on the process:

$$\dot{X}_t = b(X_t) + \sigma(X_t)\dot{W}_t$$

Differential form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

Integral form of the SDE

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$$

Lipschitz condition:

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|$$

Theorem

The SDE has a unique solution.

Proof: Picard iteration for existence

$$X_t^1 = X_0$$

$$X_t^{n+1} = X_0 + \int_0^t b(X_s^n)ds + \int_0^t \sigma(X_s^n)dB_s$$

Then,

$$X_t^{n+1} - X_t^n = \int_0^t (b(X_s^n) - b(X_s^{n-1}))ds + \int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))dB_s.$$

Define

$$f_t^n = \mathbb{E}|X_t^{n+1} - X_t^n|^2.$$

Then

$$\begin{aligned} f_t^n &\leq 2\mathbb{E}\left|\int_0^t (b(X_s^n) - b(X_s^{n-1}))ds\right|^2 \\ &\quad +2\mathbb{E}\left|\int_0^t (\sigma(X_s^n) - \sigma(X_s^{n-1}))dB_s\right|^2 \\ &\leq 2T\int_0^t \mathbb{E}|b(X_s^n) - b(X_s^{n-1})|^2 ds \\ &\quad +2\int_0^t \mathbb{E}|\sigma(X_s^n) - \sigma(X_s^{n-1})|^2 ds \\ &\leq 2K^2(T+1)\int_0^t f_s^{n-1} ds. \end{aligned}$$

Denote $L = 2K^2(T+1)$ and iterating, we get

$$\begin{aligned} f_t^n &\leq L\int_0^t L\int_0^s f_r^{n-2} dr ds \\ &= L^2\int_0^t (t-r) f_r^{n-2} dr. \end{aligned}$$

Continue this procedure, we get

$$f_t^n \leq L^n \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} f_s^1 ds.$$

This proves the converges of X^n to a process X which solves the SDE.

Uniqueness

We assume two solutions X_t and Y_t.

Similar to above, we get

$$\begin{aligned} f_t &\equiv \mathbb{E}|X_t - Y_t|^2 \\ &\leq L^n \int_0^t \frac{(t-s)^{n-2}}{(n-2)!} f_s ds \to 0. \end{aligned}$$

Thus, $X = Y$.

4. Examples

a) Geometric BM

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

Then

$$\begin{aligned} d\ln X_t &= \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 \\ &= \mu dt + \sigma dB_t - \frac{1}{2X_t^2}\sigma^2 X_t^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t. \end{aligned}$$

$$\ln X_t = \ln X_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t.$$

We get

$$X_t = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right).$$

b) Ornstein Uhlenbeck process

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t$$

Note that

$$\begin{aligned} d(e^{\theta t}X_t) &= e^{\theta t}dX_t + X_t de^{\theta t} + de^{\theta t}dX_t \\ &= e^{\theta t}(\theta(\mu - X_t)dt + \sigma dB_t) + \theta X_t e^{\theta t}dt \\ &= \theta\mu e^{\theta t}dt + \sigma e^{\theta t}dB_t \end{aligned}$$

So

$$e^{\theta t}X_t = X_0 + \int_0^t \theta\mu e^{\theta s}ds + \int_0^t \sigma e^{\theta s}dB_s.$$

Hence

$$X_t = X_0 e^{-\theta t} + \theta\mu \int_0^t e^{-\theta(t-s)}ds + \sigma \int_0^t e^{-\theta(t-s)}dB_s.$$

5. Continuous state branching process

Consider a population start with $X_0^{(n)}$ individuals. Each give birth to $2\xi_i^1$, $i = 1, 2, \cdots, X_0^{(n)}$ offsprings, where ξ_i^0 are i.i.d. Bernoulli random variable with $p = \frac{1}{2}$. Then

$$X_1^{(n)} = \sum_{i=1}^{X_0^{(n)}} 2\xi_i^0.$$

Continue this procedure, we get a discrete time MC

$$X_{k+1}^{(n)} = \sum_{i=1}^{X_k^{(n)}} 2\xi_i^k.$$

Next, we consider the scaling limit

$$\frac{1}{n} X^{(n)}_{[nt]}$$

Note that

$$X^{(n)}_{k+1} - X^{(n)}_{k} = \sum_{i=1}^{X^{(n)}_k} (2\xi^k_i - 1).$$

Summing up in k, we get

$$X^{(n)}_{[nt]} = X^{(n)}_0 + \sum_{k=0}^{[nt]-1} \sum_{i=1}^{X^{(n)}_k} (2\xi^k_i - 1).$$

Thus

$$\frac{1}{n}X_{[nt]}^{(n)} = \frac{1}{n}X_0^{(n)} + \sum_{k=0}^{[nt]-1} \sqrt{\frac{X_k^{(n)}}{n}} \frac{1}{\sqrt{X_k^{(n)}}} \sum_{i=1}^{X_k^{(n)}} (2\xi_i^k - 1)\frac{1}{\sqrt{n}}.$$

Note that

$$\frac{1}{\sqrt{X_k^{(n)}}} \sum_{i=1}^{X_k^{(n)}} (2\xi_i^k - 1)\frac{1}{\sqrt{n}} \approx N(0, \frac{1}{n}) \approx B_{\frac{k+1}{n}} - B_{\frac{k}{n}}.$$

Therefore,

$$\frac{1}{n}X_{[nt]}^{(n)} \approx \frac{1}{n}X_0^{(n)} + \sum_{k=0}^{[nt]-1} \sqrt{\frac{X_k^{(n)}}{n}} (B_{\frac{k+1}{n}} - B_{\frac{k}{n}}).$$

Taking limit, we get

$$X_t = x + \int_0^t \sqrt{X_s} dB_s$$

It is called Feller's branching diffusion.
There is no explicit solution. However, we can find its distribution through its Laplace transform

$$\mathbb{E} e^{-\lambda X_t}$$

Let T be fixed and consider ODE

$$\begin{cases} \dot{u}_t = \frac{1}{2} u_t^2, & 0 \le t \le T \\ u_T = \lambda \end{cases}$$

Then

$$u_t = \frac{2\lambda}{2 + \lambda(T - t)}$$

Next we apply Itô's formula to

$$e^{-u_t X_t}$$

$$
\begin{aligned}
& de^{-u_t X_t} \\
= \;& -u_t e^{-u_t X_t} dX_t + \frac{1}{2} u_t^2 e^{-u_t X_t} (dX_t)^2 - X_t e^{-u_t X_t} du_t \\
= \;& -u_t e^{-u_t X_t} \sqrt{X_t} dB_t + \frac{1}{2} u_t^2 e^{-u_t X_t} X_t dt - X_t e^{-u_t X_t} \dot{u}_t dt \\
= \;& -u_t e^{-u_t X_t} \sqrt{X_t} dB_t
\end{aligned}
$$

So

$$
e^{-\lambda X_T} - e^{-u_0 x} = -\int_0^T u_t e^{-u_t X_t} \sqrt{X_t} dB_t.
$$

Taking expectation, we get

$$\mathbb{E}e^{-\lambda X_T} = e^{-u_0 x} = \exp\left(-\frac{2\lambda x}{2+\lambda T}\right).$$